THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR SIRISH L. SHAH

TITLE OF THESIS THE ROLE OF EIGENVECTORS AND EIGENVALUES IN MULTIVARIABLE CONTROL SYSTEMS DESIGN

DEGREE FOR WHICH THESIS WAS PRESENTED Doctor of Philosophy

YEAR THIS DEGREE GRANTED 1977

THE UNIVERSITY OF ALBERTA

THE ROLE OF EIGENVECTORS AND EIGENVALUES IN MULTIVARIABLE CONTROL SYSTEMS DESIGN

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

IN

PROCESS CONTROL

DEPARTMENT OF CHEMICAL ENGINEERING

EDMONTON, ALBERTA

SPRING, 1977

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "The Role of Eigenvectors and Eigenvalues in Multivariable Control Systems Design", submitted by Sirish L. Shah, M.Sc., in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

This thesis is concerned primarily with an investigation of the influence of eigenvectors and eigenvalues on linear system characteristics such as disturbance rejection and eigenvalue insensitivity to system parameter variations.

Existing design procedures for eigenvector/eigenvalue assignment are briefly reviewed and the results reinterpreted using range space restrictions. Such analysis readily shows that for a n^{th} order controllable system with r inputs, complete state feedback allows arbitrary assignment of n eigenvalues and up to r elements of each of the n eigenvectors.

The importance of eigenvectors in the design of control systems is illustrated by an investigation of the 'disturbability' characteristics of multivariable systems. A linear system is defined as 'undisturbable' with respect to a particular input variable if the state or output variables of interest are not disturbed by arbitrary variations in that input. Undisturbability is closely related to the system properties of uncontrollability and structural uncontrollability, but is not identical. Necessary and sufficient conditions for $k \leq r$ state variables to be undisturbable with respect to the j^{th} disturbance, require that the system matrix (or equivalently the matrix of eigenvectors) be quasi-triangular with a kxn-k off-diagonal partition of zero elements *and* that the corresponding k elements of the j^{th} column of the input

disturbance matrix be zero. These results together with existing eigenvector/eigenvalue assignment techniques provide a simple, straightforward design procedure to produce undisturbability in linear systems. The problem of asymptotic setpoint tracking in closed-loop undisturbable systems was also considered and it is shown that a solution to this problem is almost always possible.

Feedback and feedforward controllers designed to produce undisturbability were evaluated by experimental application to a computer controlled, pilot-plant evaporator. The results were superior to conventional multiloop controllers, and comparable to controllers designed using optimal quadratic techniques. The design method was also applied to 11^{th} and 20^{th} order models of two different binary distillation columns and evaluated by digital simulation.

For the class of systems which do not satisfy the necessary and sufficient conditions for undisturbability, a design procedure is proposed that uses state feedback control to: (i) minimise the effect of external disturbances on system outputs of interest and (ii) carry out arbitrary eigenvalue assignment in the closed-loop system. An experimental evaluation of this procedure on the pilot-plant evaporator demonstrated the practicality of this approach. Using duality this procedure is also applied to the design of full order observers to minimise the effect of unmeasurable external disturbances on the state estimate of interest.

A set of simple, constructive conditions for achieving eigenvalue invariance to arbitrary variations in specified system parameters is derived in terms of the closed-loop system eigenvectors. An illustrative example demonstrates the superiority of this procedure over conventional pole placement techniques using unity-rank state feedback.

ACKNOWLEDGEMENTS

I would like to thank the following who have directly contributed in making this thesis possible:

- My supervisors D. Grant Fisher and Dale E. Seborg for their guidance, enthusiasm and understanding which contributed so much in making this thesis enjoyable.
- The National Research Council of Canada for financial support in the form of a postgraduate scholarship.
- Don Sutherland and the workshop staff, and fellow graduate students all of whom have helped to keep the evaporator operational.
- The DACS centre personnel for their technical assistance.
- Zenta Iwai, with whom I had stimulating discussions and who also proofread the text.
- Evelyn Sherwin for her excellant typing of the text.
- Finally, for love, faith and encouragement I shall always be indebted to my wife and family.

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CHAPTER 1

INTRODUCTION

The basic objective of this thesis was to investigate the effect of external disturbances on specified system state, or output variables, and to develop design procedures for controllers that would eliminate or minimise the effect of such disturbances. Most process systems operate in an environment where unknown external disturbances are almost always present and hence good regulatory control is usually a prime objective. Significant theoretical advances have been made in recent years in the control of linear multivariables systems but, very few of these new techniques have been primarily concerned with the effect of external disturbances on the overall system response. In fact, most multivariable synthesis techniques do not make full use of information available in the process model about the effect of disturbances, and many design techniques ignore disturbances altogether, or consider only special classes of disturbances such as impulses. Therefore, this study set out to look at state space models of systems and to determine under what conditions, if any, the state and/or output variables of such systems could be made invariant to arbitrary, unknown disturbances in one or

more system inputs. The main emphasis in the study was to be on the structural aspects of system models.

1.1 Relationship to Previous Work

Over the past 10 years a number of research projects have been carried out in the Department of Chemical Engineering concerned with the development and experimental evaluation of modern control techniques such as optimal multivariable feedback control, (Newell [1], Fisher and Seborg [2]); adaptive control (Oliver [3],[2]); multivariable frequency domain methods (Kuon [4]); and eigenvalue assignment by state feedback (Park [5],Topaloglu [6]). These investigations included experimental applications on a computer controlled, pilot plant evaporator which showed that most of the modern control techniques gave significantly better performance than conventional single input, single output design techniques. However, some difficulty was experienced with the experimental implementation of state feedback controllers which had been designed to assign eigenvalues to specified locations. In many cases the experimental responses were unstable and it became clear that the design procedure should impose other conditions in addition to the location of specific eigenvalues.

Therefore, one of the concerns of this investigation was the identification of additional conditions that should, and could, be incorporated into pole-placement design procedures.

It is not difficult to show from a modal analysis of the time domain response of the state variables of a system, that the eigenvectors as well as the eigenvalues, play a key role in determining the time domain response; system sensitivity; input/output decoupling; and system decomposition. In fact the eigenvalues of the closed-loop system serve only as exponential weighting factors on the terms making up the system time domain response and thus affect the form of the response. Thus, the initial direction taken in this study was to investigate the role of eigenvectors in multivariable feedback control systems and to determine what specifications could be put on the eigenvectors during the design procedure.

In addition to the theoretical studies and the development of a practical design procedure, the basic objectives of this thesis also included an experimental evaluation of the theoretical developments and design techniques.

1.2 Thesis Organisation

The main contribution of this thesis is the development of the concept of system undisturbability and the determination of necessary and sufficient conditions for a system to display this property. In general terms, system state and/or output variables are said to be undisturbable by a particular input disturbance if they are unaffected by arbitrary variations in that input. These theoretical concepts are discussed in Chapter 3 with the main emphasis being on the structural aspects of these results. The design procedure that was developed based on these theoretical results requires the use of eigenvectors/eigenvalue assignment techniques and hence this project also included the evaluation and interpretation of such techniques.

The necessary and sufficient conditions for undisturbability are not overly restrictive and are usually not difficult to satisfy. However, for those cases where they cannot be satisfied an alternative design procedure was developed that minimises the effect of external disturbances on selected state and/or output variables and also permits arbitrary eigenvalue assignment. The investigations of the structural aspects of the results for disturbance localisation also led to conditions under which specific closed-loop eigenvalues

could be made invariant to arbitrary and unknown variations in system parameters. The theoretical results and design procedures developed for disturbance localisation, disturbance minimisation and eigenvalue invariance were all evaluated by numerical simulation and by experimental application to the pilot-plant evaporator. The order of the chapters in this thesis has been designed to present the above results in a convenient order. Chapter 3 contains the main theoretical results and is a prerequisite to the full understanding of all the following chapters. However, the other chapters are relatively independent of each other and may be read in any order.

Chapter 2 presents a brief review of the main results on existing eigenvector/eigenvalue assignment techniques and conditions which specify how many and which elements of the eigenvectors can be arbitrarily assigned are given in terms of range-space restrictions. The implications of controllability on eigenvector assignment is also discussed.

In Chapter 3, the concept of undisturbability is formally defined and related to uncontrollability and structural controllability. Here the necessary and sufficient conditions for a system to have undisturbable state or output variables are expressed in

terms of the structure of the coefficient matrices of the state space model and the structure of the eigenvector matrix. These results are then used in conjunction with the eigenvector/eigenvalue assignment algorithm discussed in Chapter 2 to arrive at a design procedure for disturbance localisation.

The experimental evaluation of multivariable controllers designed to produce undisturbability is presented in Chapter 4. The performance of the disturbance localisation controllers is compared with the performance of conventional multiloop controllers and with those designed using optimal control techniques. The design procedure to achieve undisturbability is also applied to 20^{th} and 11^{th} order state space models of two different distillation columns. A discussion of the combination of proportional state feedback with integral feedback of specific state variables of interest is also included in this chapter.

In Chapter 5, a method is presented for designing state feedback controllers which assign eigenvalues of the closed-loop system to specified positions, and also minimise the effect of external disturbances on state variables of interest. Results from the experimental application of this method to the double-effect evaporator are also included. An application of the

method to the design of full order observers for systems with unmeasurable disturbances, is also included.

In Chapter 6, a method is presented for designing constant multivariable feedback controllers that make selected closed-loop eigenvalues invariant to unknown perturbations of arbitrary magnitude in system parameters. The sufficient conditions for eigenvalue invariance are expressed in terms of the structure of the closed-loop system matrices.

Some overall conclusions and recommendations for future work are presented in Chapter 7.

Appendix A is devoted to the discussion of selected geometric concepts which are used in the main body of the thesis. Appendix B outlines in detail the difficulties associated with eigenvector assignment using output feedback. Appendix C contains a description of the pilot-plant double effect evaporator; the associated instrumentation; and the computer equipment. The relationship between the concepts of invariant subspaces, invariant zeros and parametric sensitivity is discussed in Appendix D. A new algorithm to compute the invariant zeros of a system is also presented in this Appendix.

CHAPTER 2

EIGENVALUE/EIGENVECTOR ASSIGNMENT FOR MULTIVARIABLE SYSTEMS

Abstract

Existing design procedures for eigenvalue/eigenvector assignment are briefly reviewed. The problem of eigenvalue/eigenvector assignment using state or output feedback is considered using range space restrictions. This readily shows how many and which elements of the closed-loop system eigenvector or reciprocal basis vector can be specified. The implications of controllability in eigenvector assignment is also discussed.

2.1 Introduction

Since 1962 when Rosenbrock [1] first introduced modal control as a possible design aid in the control of large chemical plants, considerable attention has been paid to control system design by assigning the eigenvalues of the closed-loop system. Eigenvalue assignment techniques have been widely used in the control system design literature to improve system stability and the speed of response of interacting processes. A wide variety of different algorithms/methods exist for eigenvalue placement via state feedback. Most of the well-known algorithms and results in this area, prior to and around 1973, have been summarised in surveys by Topaloglu [2] and Park [3].

It is well known that for a linear, controllable system, eigenvalues can be assigned to arbitrary positions by using state feedback. Conventionally this has been achieved either by using unity-rank state feedback (simultaneous design) or by non-unity rank state feedback (recursive design). However, for multi-input systems the controller is not unique since more than one feedback controller can be designed to achieve the desired eigenvalue placement. Different feedback controllers result in different system performance. Closed-loop system

eigenvectors are equally important in determining system performance. This becomes evident at once from a modal analysis of the closed-loop system.

For an n^{th} order controllable system with r inputs, only n degrees of freedom are required to assign all n eigenvalues to arbitrary locations. The remaining $n \times (r-1)$ degrees of freedom available in the design of a state feedback controller can be utilized to assign components of the eigenvectors. However, very few algorithms/methods are available at the present time that fully utilise all degrees of freedom available to arbitrarily assign some elements of the eigenvectors as well as all of the closed-loop system eigenvalues.

Srinathkumar and Rhoten [4] have proposed an algorithm for the computation of a state feedback matrix that assigns closed-loop eigenvalues as well as certain elements of the closed-loop eigenvectors. The relationship between their algorithm and the matrix pseudoinverse approach was shown as part of this work (cf. Section 2.2 and Shah et al. [5]). In [5], application of simple geometric concepts also showed how many and which elements of the closed-loop eigenvectors can be assigned arbitrarily. More recently Moore [6] has provided a simple, but general, mathematical characterization of the class of all closed-loop eigenvector sets which can be obtained

for a given set of distinct closed-loop eigenvalues. Using a minimisation procedure, Ramar and Gourishankar [7] have also proposed a method whereby all the closed-loop system eigenvalues and some or all of the elements of the closed-loop eigenvector matrix can be assigned. In deriving an alternative proof for conditions for pole assignability, Kimura [8] has also shown how eigenvalues and eigenvectors can be assigned using state feedback control.

Eigenvalue/eigenvector assignment procedures are discussed here because they form an important part of the design procedures developed in later chapters. The present chapter is organized as follows. The problem of eigenvalue/eigenvector assignment using state feedback is considered in Section 2.2. The implications of controllability in eigenvector assignment is discussed in Section 2.3. The difficulties associated with eigenvalue/eigenvector assignment using incomplete state feedback is considered in Section 2.4 followed by some concluding remarks in Section 2.5.

2.2 Assignment of Closed-loop Eigenvalues and Eigenvectors

Consider the n^{th} order completely controllable, linear, time-invariant system.

$$\dot{x}(t) = A\ x(t) + B\ u(t)$$
$$y(t) = C\ x(t) \tag{2.1}$$

with r inputs, m outputs and a state feedback control law

$$u(t) = K\ x(t) \tag{2.2}$$

The resulting closed-loop system matrix, H, can be expressed as

$$H = A + BK = WJV \tag{2.3}$$

where J is the Jordan canonical form defined by the desired closed-loop eigenvalues; W is the closed-loop system eigenvector matrix; and $V=W^{-1}$ is the matrix of reciprocal basis vectors. The following relationships apply when the closed-loop system matrix, H or WJV, is fully specified

i) A control matrix K which exactly satisfies eqn. (2.3) exists if and only if

$$\text{Range } (WJV-A) \subseteq \text{Range } B \tag{2.4}$$

ii) If condition (2.4) is satisfied, then a control matrix K, which exactly satisfies eqn (2.3), can be computed from

$$K = B^{*}(WJV-A) \tag{2.5}$$

where B^* denotes the pseudoinverse of B given by $B^* = (B^TB)^{-1}B^T$ and T denotes the transpose of a matrix.

The manner in which condition (2.4) determines how many, and which, elements of W can be specified arbitrarily is illustrated by writing eqn. (2.3) in the partitioned form:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] \left[\begin{array}{cc} K_1 & K_2 \end{array}\right] = \left[\begin{array}{c|c} W_{11} & W_{12} \\ \hline W_{21} & W_{22} \end{array}\right] \left[\begin{array}{c|c} J_1 & 0 \\ \hline 0 & J_2 \end{array}\right] \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{21} & V_{22} \end{array}\right] \tag{2.6}$$

where A_{11}, B_1, K_1, W_{11}, J_1 and V_{11} are rxr matrices, A_{22}, W_{22}, J_2 and V_{22} are (n-r)x(n-r) matrices and B is assumed to have full rank. From the above eqn. (2.6) it is not difficult to see that the rxr entries in W_{11} and the rx(n-r) entries in W_{12} can be chosen to be completely arbitrary if and only if Range $(B_1) = R^r$, i.e. if and only if B_1 is nonsingular. Implicit in the above condition is the requirement, since $V = W^{-1}$, that W be nonsingular. This in turn implies that the closed-loop system should have distinct eigenvalues. Such a restriction, however, is not limiting. Note that specifying the first r elements of each of the n eigenvectors requires only rxn-n degrees

of freedom, i.e. rxn-n elements of K, providing that at least one of the r specified values in each eigenvector is nonzero. Once W_{11} and W_{12} are so chosen, W_{21} and W_{22} are computed such that condition (2.4) is satisfied.

One method of calculating W_{21} and W_{22} is to use the algorithm presented by Srinathkumar and Rhoten [4]. They start with an equation identical to eqn. (2.6) and by algebraic manipulation determine the degrees of freedom available for specifying elements of W. The algorithm uses state feedback control to assign the closed-loop eigenvalues as well as elements of the eigenvector matrix. The derivation of the algorithm requires some tedious algebraic manipulations, as shown in Appendix B. The two main equations of the algorithm are:

$$W_{21}J_1 - PW_{21} = TW_{11} + SW_{11}J_1 \qquad (2.7)$$

$$W_{22}J_2 - PW_{22} = TW_{12} + SW_{12}J_2 \qquad (2.8)$$

where

$$S = B_2B_1^{-1}, \quad T = A_{21} - SA_{11} \quad \text{and} \quad P = A_{22} - SA_{12}.$$

When the right hand terms in eqns. (2.7) and (2.8) are known, the algorithm represents a set of n^2-nr linear equations in as many unknowns. It can be shown that, if W_{11} and W_{12} are chosen arbitrarily and W_{21} and W_{22} are calculated by the above algorithm, then the resulting W

will satisfy condition (2.4). It therefore follows that K can be computed using the matrix pseudoinverse approach as in eqn. (2.5). Note that an identical K can also be evaluated from eqn. (2.6). There is a computational advantage in using the pseudoinverse approach, that is it is applicable generally. For example, when condition (2.4) is not satisfied exactly due to numerical round-off errors, no control matrix can satisfy eqn. (2.3). However, the control matrix computed from eqn. (2.5), using the pseudoinverse of B, satisfies eqn. (2.3) in the least-squares sense.

Moore [6] has characterised the class of all closed-loop eigenvector sets which can be obtained for a given set of distinct closed-loop eigenvalues using state feedback. This is a more general result that does not require B_1 to be nonsingular. His result requires that if, for a given closed-loop eigenvalue λ_1,

$$[\lambda_i I-A \mid B] \begin{bmatrix} N_{\lambda_i} \\ \hline M_{\lambda_i} \end{bmatrix} = 0 \qquad (2.9)$$

Where the column vectors of $[N_{\lambda_i} \; M_{\lambda_i}]^T$ span Ker $[\lambda_i I-A \mid B]$. Then for this system only those eigenvectors, w_i, corresponding to λ_i can be assigned such that

i) $w_i \in \text{span}\,(N_{\lambda_i})$

ii) for $i \in n$ vectors w_i are linearly independent vectors in $\mathbb{C}^n$, and

iii) $w_i = \bar{w}_j$ whenever $\lambda_i = \bar{\lambda}_j$, where $^-$ denotes a complex conjugate.

In [6] Moore has made some remarks regarding controllability and eigenvector assignment. While equivalence of controllability and pole assignability is well known, the relationship between controllability and eigenvector assignment, however, is not so well known. The following section contains a discussion of controllability considerations in eigenvector assignment based on the algorithm in [4].

2.3 Controllability Considerations in Eigenvector Assignment

Consider an n^{th} order, r-input uncontrollable system. Let the Kalman canonical form of such a system be characterised by the pair $(\tilde{A},\tilde{B})$, where matrices $\tilde{A}$ and $\tilde{B}$ in their partitioned form are:

$$\left[\begin{array}{c|c:c} \tilde{A}_{11} & \tilde{A}^1_{12} & \tilde{A}^2_{12} \\ \hline \tilde{A}^1_{21} & \tilde{A}^1_{22} & \tilde{A}^2_{22} \\ \hdashline 0 & 0 & \tilde{A}^4_{22} \end{array}\right], \quad \left[\begin{array}{c} \tilde{B}_1 \\ \hline \tilde{B}^1_2 \\ \hdashline 0 \end{array}\right] \qquad (2.10)$$

In eqn. (2.10), $\tilde{A}_{11} \in R^{rxr}$, $\tilde{B}_1 \in R^{rxr}$, $\tilde{A}^1_{21}$ and $\tilde{B}^1_2 \in R^{\ell xr}$, $\tilde{A}^1_{22} \in R^{\ell x(n-r-\ell)}$ and $\tilde{A}^4_{22} \in R^{\ell x\ell}$. To relate the partitioned form for the pair $(\tilde{A}\ \tilde{B})$ in eqns. (2.6) and (2.10) it should be observed that:

$$\tilde{A}_{12} = \left[\begin{array}{c|c} \tilde{A}^1_{12} & \tilde{A}^2_{12} \end{array}\right], \quad \tilde{A}_{21} = \left[\begin{array}{c} \tilde{A}^1_{21} \\ \hline 0 \end{array}\right], \quad \text{and } \tilde{A}_{22} = \left[\begin{array}{c|c} \tilde{A}^1_{22} & \tilde{A}^2_{22} \\ \hline 0 & \tilde{A}^4_{22} \end{array}\right]$$

From eqn. (2.10) it is clear that the system under consideration is uncontrollable and that the eigenvalues of $\tilde{A}^4_{22}$ correspond to the uncontrollable modes of the system. Application of the eigenvector/eigenvalue assignment algorithm, i.e. eqns. (2.7) and (2.8) from [4] to the pair $(\tilde{A},\tilde{B})$ as shown in (2.10) leads to the following results.

$$S = \begin{bmatrix} \tilde{B}^1_2\tilde{B}_1^{-1} \\ 0 \end{bmatrix} \text{ which can be rewritten as } \equiv \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} \tilde{A}^1_{21} \\ 0 \end{bmatrix} - \begin{bmatrix} S_1 \\ 0 \end{bmatrix} \tilde{A}_{11} = \begin{bmatrix} \tilde{A}^{-1}_{21} - S_1\tilde{A}_{11} \\ 0 \end{bmatrix} \equiv \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \qquad 2.11$$

$$P = \begin{bmatrix} \tilde{A}^1_{22} & \tilde{A}^2_{22} \\ 0 & \tilde{A}^4_{22} \end{bmatrix} - \begin{bmatrix} S_1 \\ 0 \end{bmatrix} \tilde{A}_{12} = \begin{bmatrix} P_1 & P_2 \\ 0 & \tilde{A}^4_{22} \end{bmatrix}$$

where matrices S,T and P have been defined in eqn. (2.8). Since the primary concern here is the implications of

assigning eigenvectors corresponding to uncontrollable modes, eqn. (2.8) will now be considered. For an arbitrary choice of W_{12} and J_2 given by

$$W_{12} = \begin{bmatrix} W_{12}^1 & W_{12}^2 \\ W_{12}^3 & W_{12}^4 \end{bmatrix}, \quad J_2 = \begin{bmatrix} J_2^1 & 0 \\ 0 & J_2^2 \end{bmatrix}$$

where W_{12} and J_2 have been compatibly partitioned with respect to matrices T and S, the appropriate elements of W_{22} are given by:

$$\left[\begin{array}{c|c} W_{22}^1 J_2^1 - P_1 W_{22}^1 - P_2 W_{22}^3 & W_{22}^2 J_2^2 - P_1 W_{22}^2 - P_2 W_{22}^4 \\ \hline W_{22}^3 J_2^1 - \tilde{A}_{22}^4 W_{22}^3 & W_{22}^4 J_2^2 - \tilde{A}_{22}^4 W_{22}^4 \end{array}\right] =$$

$$\left[\begin{array}{c|c} T_1 W_{12}^1 + T_2 W_{12}^3 + S_1 W_{12}^1 J_2^1 + S_2 W_{12}^3 J_2^1 & T_1 W_{12}^2 + T_2 W_{12}^4 + S W_{12}^2 J_2^2 + S W_{12}^4 J_2^2 \\ \hline 0 & 0 \end{array}\right] \tag{2.12}$$

Eqn. (2.12) clearly implies that diag $(J_2^2) = \sigma(\tilde{A}_{22}^4)$ and that W_{22}^4 is the eigenvector matrix of $\tilde{A}_{22}^4$. Since distinct eigenvalues have been assumed to have been specified, eqn. (2.12) also implies that $W_{22}^3 = 0$, which is intuitively expected. This leaves two matrix linear equations to be solved for W_{22}^1 and W_{22}^2 which can be solved for any arbitrary value of W_{12}. (Note that if

one or more vectors in W_{12} are selected to be zero then the corresponding eigenvalue in J_2 should be the appropriate eigenvalue of P). The conclusion to be drawn from the above analysis is that the r elements of each of the eigenvectors i.e. W_{12}^2 and W_{12}^4 corresponding to the uncontrollable modes can be arbitrarily assigned, while the corresponding ℓ eigenvalues remain invariant under feedback and thus they should be specified in the J_2^2 matrix block to be identical to the eigenvalues of $\tilde{A}_{22}^4$.

The main point to note here is that while controllability is necessary and sufficient for pole assignment it is not necessary for eigenvector assignment (i.e. for assigning r elements in each eigenvector).

2.4 <u>Eigenvalue/Eigenvector Assignment Using Output Feedback</u>

Consider eqn. (2.1) in conjunction with the output feedback control law,

$$u(t) = Fy(t) = FCx(t) \tag{2.13}$$

the resulting closed-loop system matrix is given by

$$(A + BFC) = WJV \tag{2.14}$$

For a given closed-loop system matrix, WJV, the appropriate F exists if and only if condition (2.4) and the following condition are satisfied:

$$\text{Range } (WJV - A)^T \subseteq \text{Range } C^T \qquad (2.15)$$

conditions (2.4) and (2.15) impose significant restrictions on the specification of the elements of the closed-loop system eigenvectors and reciprocal basis vectors, and hence in the general case a number of difficulties arise in the design of F. These are discussed at the end of this section. Normally, n of the rxm available degrees of freedom (assuming rxm > n) are used to assign the closed-loop eigenvalues, leaving only rxm-n degrees of freedom available for assigning elements of the eigenvectors or of the reciprocal basis vectors. Thus, in the general case, it is probably not worthwhile to attempt to assign elements of the eigenvectors or reciprocal basis vectors unless rxm>>n.

The output-feedback design problem for the special case of an n-input, n-state, m-output ($m \leq n$) system can be readily handled because condition (2.4) is always satisfied for such systems, and hence only condition (2.15) need be considered for the design procedure. The algorithm for this case is the dual (or adjoint) of the one for state feedback control and can

be obtained by using algebraic manipulations on the dual of the system of eqn. (2.1). Alternatively, it can also be derived by considering the following partitioned form of eqn. (2.14)

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] F \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] =$$

$$\left[\begin{array}{c|c} W_{11} & W_{12} \\ \hline W_{21} & W_{22} \end{array}\right] \left[\begin{array}{c|c} J_1 & 0 \\ \hline 0 & J_2 \end{array}\right] \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{21} & V_{22} \end{array}\right] \tag{2.16}$$

Where A_{11}, C_1, W_{11}, J_1 and V_{11} are now mxm matrices; A_{22}, W_{22}, J_2 and V_{22} are (n-m) x (n-m) matrices; B_1 is an mxr matrix; F is an rxm matrix; and C_1 and B are assumed nonsingular, obtained, if necessary, by reordering of the state variables. Bearing in mind that Range $C_1^T = R^m$, it follows that the mxm entries in V_{11} and the (n-m)xm entries in V_{21} can be chosen arbitrarily, subject only to the requirement, since $W = V^{-1}$, that V be nonsingular. Once V_{11} and V_{21} are chosen, V_{12} and V_{22} can be computed such that condition (2.15) is satisfied. Algebraic manipulation of eqn. (2.16) together with the use of the identity VW = I leads to the following two equations for calculating V_{12} and V_{22}:

$$J_1V_{12} - V_{12}R = J_1V_{11}Q + V_{11}M \qquad (2.17)$$

$$J_2V_{22} - V_{22}R = J_2V_{21}Q + V_{21}M \qquad (2.18)$$

where $Q = C_1^{-1}C_2$, $M = A_{12} - A_{11}Q$, $R = A_{22} - A_{21}Q$.

The derivation of these equations is discussed in detail in Appendix B. Eqns. (2.17) and (2.18) are linear in V_{12} and V_{22} and permit J_1, J_2, V_{11} and V_{21} to be specified. It can also be shown that V_{12} and V_{22} obtained from eqns. (2.17) and (2.18) satisfy condition (2.15) and hence this is a practical approach for designing output feedback controllers for n-input, n-state systems.

The output feedback matrix can also be computed using the pseudoinverse approach. If condition (2.15) holds the following equation leads to the same output feedback matrix as eqn. (2.16):

$$F = B^{-1}(WJV - A)C^* \qquad (2.19)$$

where $C^* = C^T(CC^T)^{-1}$ denotes the pseudoinverse of C.

The restrictions that conditions (2.4) and (2.15) impose on the general output-feedback design problem for $r<n$ for eigenvalue/eigenvector assignment can now be

realized by considering the following steps in the design procedure, further details of which are discussed in Appendix B.

i) Arbitrary choice of V_{11}, V_{21}, W_{11} and W_{12} is made subject to the constraint that the identity: $W_{11}V_{11} + W_{12}V_{12} \equiv I_m$ (from $WV = I$) be preserved.

ii) After V_{11}, V_{21}, W_{11} and W_{12} have been specified, V_{12} and V_{22} can be calculated from eqns. (2.17) and (2.18) (to satisfy condition (2.15)) and W_{21} and W_{22} calculated from the dual form of eqns. (2.17) and (2.18) i.e. eqns. (2.7) and (2.8).

iii) With V and W determined, a check must be made to determine if the identity $WV = I$ holds. If not, the entire procedure is repeated with a different choice of V_{11}, V_{21}, W_{11} and W_{12}.

Implicit in the above mentioned procedure is the assumption that the eigenvalues of the closed-loop system can be arbitrarily assigned. This usually implies that $(r+m-1)\geq n$, Topaloglu and Seborg [9]. If arbitrary assignment of all eigenvalues is not assumed then the algorithm would become prohibitively complex. Therefore in general, unless $r\times m \gg n$ (as remarked earlier) and $(r+m-1)\geq n$, it is probably not worthwhile to attempt to carry out eigenvalue/eigenvector assignment using output feedback.

The problem of eigenvalue/eigenvector assignment using incomplete state feedback can also be considered by noting that the i^{th} column, k_i of the state feedback matrix, K, can be expressed as a function of h_i and a_i, the i^{th} columns of H and A respectively, by:

$$k_i = B^*(h_i - a_i) \tag{2.20}$$

It is easily observed from eqn. (2.20) that if measurement of x_i is not available then one possible way of making $k_i = 0$ is to have $h_i = a_i$. However, it is not always easy to choose W_{11} and W_{12} such that the resulting $h_i = (WJV)_i = a_i$. There are some exceptions to this case and these will be discussed in the application of the eigenvalue/eigenvector assignment algorithm to the distillation column example in Chapter 4, where eqn. (2.20) is used to advantage in reducing the number of state variables required for feedback control.

2.5 Conclusions

The problem of eigenvalue/eigenvector assignment using state or output feedback has been considered using range space restrictions. For the state feedback case such an analysis readily shows how many and which elements of the closed-loop system eigenvector can be specified. The implications of

controllability on eigenvector assignment are discussed and it is shown that in using the algorithm in [4], controllability is not necessary for eigenvector assignment. An eigenvalue/eigenvector assignment method for output feedback control of n^{th} order, n-input, m-output ($m \le n$) systems is also presented. The restricted set of range space conditions and the associated difficulties in eigenvector/eigenvalue assignment for a general (n^{th} order, r-input, m-output) system using output feedback are discussed.

CHAPTER 3

UNDISTURBABILITY OF LINEAR MULTIVARIABLE SYSTEMS

Abstract

The concept of undisturbability, is formally defined and related to uncontrollability and structural controllability. Necessary and sufficient conditions for a system to have undisturbable state or output variables are expressed in terms of the structure of the coefficient matrices in the state-space model and the structure of the eigenvector matrix. These results and the eigenvector/ eigenvalue assignment algorithm,as discussed in Chapter 2, provide the basis of a design procedure for synthesizing multivariable controllers which achieve disturbance loc-alisation.

3.1 Introduction

Practical control systems must operate in environments where unknown external disturbances are invariably present. Some typical examples of such disturbances in control problems are: load fluctuations in power systems; ambient temperature changes and unexpected variations in feed conditions in distillation columns, chemical reactors, etc.; wind gusts and turbulence acting on aircrafts, etc.. Despite the inevitability of such external disturbances, synthesis techniques for multivariable control systems usually ignore them or consider only a special class of disturbances such as impulses. Consequently, the resulting multivariable controllers may not perform satisfactorily in the presence of other types of disturbances, particularly sustained or persistently fluctuating disturbances.

In this chapter a linear system is said to be "undisturbable" with respect to a particular input variable if the state or output variable(s) of interest are not disturbed by arbitrary variations in that input. The property of undisturbability is of considerable interest to control engineers because it implies perfect regulatory control in the sense that the input disturbance does not affect the system variable(s) of interest in any way.

In the present investigation, the problem of controlling systems subjected to persistent, unknown external disturbances is considered from a system structure point of view. Necessary and sufficient conditions for an open-loop or closed-loop system to have undisturbable state or output variables are expressed in terms of the structure of the state space matrices. An alternative set of necessary and sufficient conditions for undisturbability are also derived in terms of the closed-loop system eigenvectors. These latter conditions and the eigenvector/eigenvalue assignment algorithm considered in Chapter 2 provide a design basis for synthesizing multivariable controllers which produce disturbance localisation.

These results are equivalent to existing necessary and sufficient conditions on disturbance localisation derived from a geometrical characterization of the problem. However, in contrast to previously known geometric results, the necessary and sufficient conditions developed here are easier and more straightforward to use, both from a computational, as well as, applications point of view. In contrast, the geometric results that already exist do not lend themselves easily to computer implementation; furthermore, the computational operations that they require are subject to the hazards of numerical ill-conditioning.

3.2 Previous Work

The problem of controlling systems subjected to persistent, unknown, external disturbances, i.e. the regulatory control problem has been one of the oldest and most fundamental problems of control. The regulatory control problem has been widely researched. There is a variety of terms used in the literature to describe this and related problems. The main terms that will appear in the discussion below are: *disturbance localisation*, *disturbance rejection* and system *invariance*.

Rutman and Epelman [1] were amongst the first researchers to carry out a formal treatment of this subject in their study of the effect of external disturbances on the output of a system. They defined a system output to be invariant with respect to a disturbance if the output was completely independent of the disturbance, and they obtained necessary and sufficient conditions for a single-input, single-output system to be invariant. Preminger and Rootenberg [2] defined a system to be steady state invariant or completely invariant to a class of disturbances depending on whether the steady state or the entire effect of the disturbances is eliminated. They also discussed, heuristically, the design of compensators to attain complete invariance. Using variational methods Rozoner [3] also obtained necessary and

sufficient conditions for a single-input, single-output system to be invariant. By using Rozoner's result, Wang [4] related the concept of invariance to uncontrollability and unobservability. Later Cruz and Perkins [5,6] generalized the results on invariance to the multivariable case and also obtained results relating invariance to unobservability.

As early as 1966, C.D. Johnson [7] conceived the idea of invariant hyperplanes and discussed their role in state space theory. Geometrically, a hyperplane passing through the origin is a subspace. Johnson's results were actually based on a series of earlier Soviet publications. It was a geometric concept related to this, namely that of (A,B)-invariant subspaces that finally led to general results on disturbance localisation. In a later publication, Johnson [8] considered regulation of a linear system subjected to constant disturbances using an optimal control formulation. In a series of papers [9,10], he extended these results to a class of unknown disturbances that satisfy a specified linear differential equation, or that can be characterized by certain waveform types or modes.

Basile and Marro [11] and Wonham and Morse [12] independently introduced the concept of (A,B)-invariant

subspace and (A,B)-controllability subspace. These concepts played a basic role in the derivation of a general result on disturbance localisation by Wonham and Morse [12]. The main result of Wonham and Morse is a sufficient and necessary condition that is also a synthesis result for disturbance localisation using state feedback control. By considering the problem of zeroing the disturbance transfer function in a linear system, Bhattacharyya [13] has also derived a set of equivalent geometric necessary and sufficient conditions for disturbance localisation. Using an algebraic approach Shah et al. [14] - (cf. Section 3.5) have also derived a set of necessary and sufficient conditions for disturbance localisation. In contrast to the synthesis result in [12] the results in [13,14] are structural results, and as would be expected they are weaker than the conditions given in [12]. In an earlier investigation McLane and Davison [15] have also reported algebraic conditions which are sufficient for disturbance localisation. An extensive treatment of the regulator problem based on an algebraic approach can also be found in Silverman and Payne [16].

In a more recent paper Fabian and Wonham [17] have reported necessary and sufficient conditions for simultaneous disturbance localisation and input-output decoupling. They have also proved conditions for disturbance

rejection and closed-loop stability. Using a geometric approach, Chang and Rhodes [18] have also derived necessary and sufficient conditions for simultaneous disturbance localisation and pole assignment or closed-loop stability.

In [13] Bhattacharyya derived solvability conditions for disturbance localisation using feedback plus feedforward control. Shah [19] (cf. Section 3.6) has derived necessary and sufficient conditions for the existence of a feedforward controller that would localise the effect of external disturbances. More recently Bhattacharyya [20] has described a design procedure for designing a disturbance localisation controller that uses dynamic feedback and feedforward compensators.

An alternative design strategy is to design a controller which reduces the effects of disturbances on selected state or output variables. Willems [21] and Muller and Luckel [22] have recently proposed synthesis procedures for the design of state feedback controllers that minimise or reduce the effect of disturbances on outputs of interest. Chapter 5 presents a similar procedure that minimises the effects of disturbances on outputs of interests and also allows arbitrary pole assignment.

By a geometrical characterization of the regulator

problem, Bhattacharyya et al. [23] have derived conditions under which a zero output of a system is maintained at zero, and any non-zero output due to non-zero initial conditions or impulse-type disturbances is made to approach zero. Davison [24] and Grasselli and Nicolo [25] have also considered the design of output feedback controllers to accommodate a class of disturbances that can be described by a set of differential equations. Johnson's work [8,9,10] which was discussed earlier, also falls in this category except that he requires state feedback.

3.3 Statement of the Problem

Consider the linear time-invariant dynamic system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + D\xi(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.1}$$

where $x \in \mathcal{R}^n$ is the state, $u \in \mathcal{R}^r$ is the input, $\xi \in \mathcal{R}^q$ is the disturbance and $y \in \mathcal{R}^m$ is the output; A, B, C and D are constant matrices of appropriate dimensions. If a feedback-feedforward control system of the form

$$u(t) = Kx(t) + K^{FF}\,\xi(t) \tag{3.2}$$

is employed, the resulting closed-loop system is

$$\dot{x}(t) = Hx(t) + L\xi(t) \tag{3.3}$$

where H and L are defined by

$$H = A + BK \tag{3.4}$$

and

$$L = D + BK^{FF} \tag{3.5}$$

The disturbance localisation problem can now be posed as follows: For the closed-loop system in eqn. (3.3), is it possible to decouple the effect of $\xi(t)$ from the system outputs (or some of the state variables), for all $t > 0$ by an appropriate choice of K (or K and K^{FF} if feedforward control is possible)?

3.4 Definitions

The time response, $x(t)$, of the system in eqn. (3.3) can be expressed as the sum of two terms: $x^o(t)$, due to the initial state $x(0)$; and $x^\xi(t)$, due to the disturbances, i.e.

$$x(t) = x^o(t) + x^\xi(t) \tag{3.6}$$

where $x^o(t) = \exp(Ht)x(0)$ and

$$x^\xi(t) = \int_o^t \exp\{H(t-\tau)\}L\xi(\tau)d\tau \tag{3.7}$$

In this investigation, the term, $x^\xi(t)$, is the primary

concern. For simplicity, the disturbance rejection characteristics of system (3.3) will be considered in terms of a single disturbance, ξ_j, $j \in (1,2,\ldots,q)$. However, the results that are obtained can be easily generalized to all q disturbances. It is clear from eqn. (3.7) that the disturbance rejection characteristics of system (3.3) with respect to disturbance, ξ_j, are characterised by the pair (H,ℓ_j) for the state variables, and by (H,ℓ_j,c_i^T) for the i^{th} output where ℓ_j is the j^{th} column of L and c_i^T is the i^{th} row of C.

Definition 1: *State Undisturbability*

The i^{th} state variable, x_i, of a system characterised by the pair (H,ℓ_j) is said to be undisturbable with respect to disturbance, ξ_j, if for arbitrary $\xi_j(t)$, and for all $t > 0$, x_i^ξ, the i^{th} element of x^ξ, satisfies:

$$x_i^\xi(t) = 0 \tag{3.8}$$

Definition 2: *Output Undisturbability*

The i^{th} output, y_i, of a system characterised by (H,ℓ_j,c_i^T) where c_i^T is the i^{th} row of C is said to be undisturbable with respect to disturbance, ξ_j, if for arbitrary $\xi_j(t)$, and for all $t > 0$, y_i^ξ, the i^{th} element of y^ξ, satisfies

$$y_i^\xi(t) = 0 \tag{3.9}$$

When condition (3.8) (or (3.9)) is not satisfied for some $\xi_j(t) \neq 0$ and some $t > 0$, then the i^{th} state variable (or output variable) is said to be *disturbable*. When all elements of the state (or output) vector are undisturbable or disturbable, the modifier complete is used; for example, as in complete state (or output) disturbability. Complete state undisturbability implies that the disturbance has no effect on the system, as would be the case with a "perfect" control policy.

The concept of undisturbability is closely related to the concepts of uncontrollability and structural controllability, as will be discussed in Section 3.10.

3.4.1 Undisturbability in terms of signal flowgraphs

State undisturbability of linear systems can also be considered in terms of a signal flowgraph formulation. Consider the following third order open-loop system with the pair (A, d_j) in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad d_j = \begin{bmatrix} 0 \\ 0 \\ d_{3j} \end{bmatrix} \qquad (3.10)$$

where d_{3j} and all elements of A are nonzero except for a_{13} and a_{23}. The signal flowgraph of the system characterized by the pair (3.10) is shown in Figure 3.1.

In signal flowgraph terminology, the state variable, x_i, is *nonaccessible* to ξ_j, if and only if there is no possibility of reaching x_i starting from ξ_j and proceeding in the direction of the arrows along any of the paths in the signal flowgraph of the pair (A,d_j). Similar considerations for the closed-loop system, (H,ℓ_j) lead to the following equivalent definition of state undisturbability.

Definition 3: *State Undisturbability in Signal Flowgraph Terminology:*

The i^{th} state variable, x_i, of a system characterised by the pair (H,ℓ_j), is said to be undisturbable with respect to disturbance, ξ_j, if x_i is nonaccessible to ξ_j.

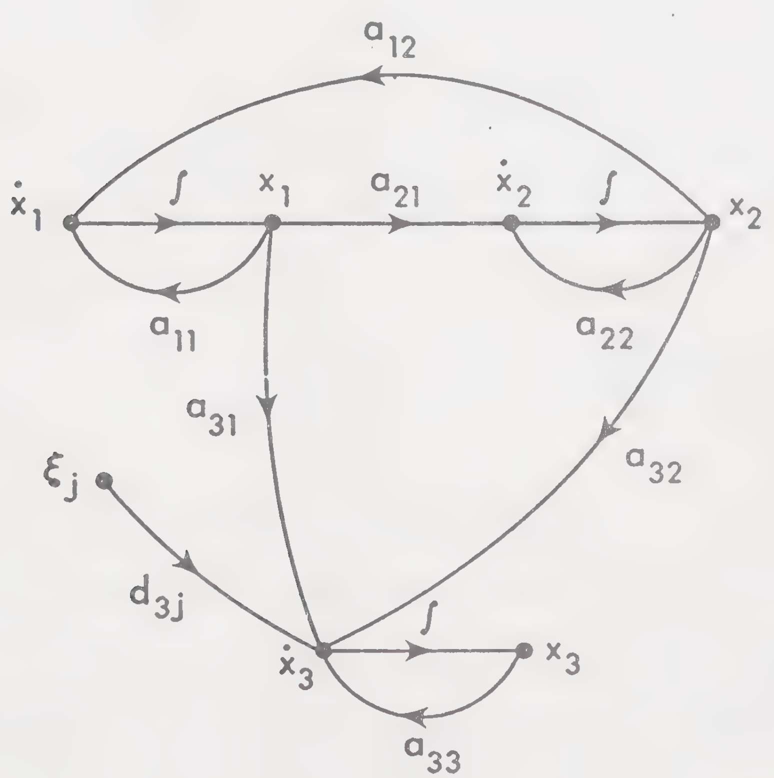

Fig. 3.1 Signal flowgraph of the pair (A,d_j) as in eqn. (3.10).

3.5 Necessary and Sufficient Conditions for Output Undisturbability Using an Algebraic Approach.

The necessary and sufficient condition of Wonham and Morse [12] for disturbance localisation using state feedback control is based on the concept of an (A,B)-invariant subspace. This condition is as follows:

$$\mathcal{D} \subset \mathcal{V}^* \tag{3.11}$$

where $\mathcal{D}$ denotes the range space of D and $\mathcal{V}^*$ is the maximal (A,B)-invariant subspace contained in Kernel C.

It is well known that the (A,B)-invariant subspace is spanned by the eigenvectors of the closed-loop system, and hence it is possible to express the necessary and sufficient conditions for disturbance localisation (or decoupling) in terms of the closed-loop system eigenvectors.

Assuming distinct eigenvalues for the closed-loop system, the output $y^{\xi}(t)$ of the system of eqn. (3.3), can be represented by

$$y^{\xi}(t) = C\int_{0}^{t} [W \exp\{\Lambda(t-\tau)\}V]D\xi(\tau)\, d\tau \tag{3.12}$$

where $\qquad W \exp\{\Lambda(t-\tau)\}V = \exp\{H(t-\tau)\}$

that is, W is the closed-loop system eigenvector matrix, $W = V^{-1}$, and since distinct eigenvalues have been assumed Λ is a diagonal matrix whose diagonal elements are the closed-loop system eigenvalues.

Now define the m x q matrix $M(\tau)$ by

$$M(\tau) = CW\,\exp\{\Lambda(t-\tau)\}VD \tag{3.13}$$

It is then clear from eqn. (3.12) that the necessary and sufficient condition for output undisturbability is

$$M(\tau) = 0 \quad \text{for} \quad \tau > 0 \tag{3.14}$$

In terms of element $m_{kj}(\tau)$ of $M(\tau)$, the condition of eqn. (3.14) becomes

$$m_{kj}(\tau) = \sum_{i=1}^{n} c_k^T w_i v_i^T d_j \exp\{\lambda_i(t-\tau)\} = 0 \tag{3.15}$$

where the superscript T denotes the transpose of a vector, c_k^T is the k^{th} row of C, w_i and v_i are the i^{th} eigenvector and reciprocal eigenvector, respectively; d_j is the j^{th} column of D and λ_i is the i^{th} eigenvalue. Assuming that the eigenvalues are distinct, and that each element of $M(\tau)$ is equal to zero for all $\tau > 0$, it is easy to show that the necessary and sufficient condition expressed by eqn. (3.15) is equivalent to

$$c_k^T w_i v_i^T d_j = 0 \qquad k \in m,\ i \in n,\ j \in q \tag{3.16}$$

where $k = 1,2,\ldots,m$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,q$ are abbreviated as $k \in m$, $i \in n$ and $j \in q$, respectively. Using eqn. (3.16), the following result is stated in the

form of a theorem:

Theorem 3.1

The condition $c_k^T w_i v_i^T d_j = 0$, $k \in m$, $i \in n$, $j \in q$ *can be satisfied if and only if* $\mathcal{D} \subset \mathcal{V} \subset \text{Ker } C$, *where* $\mathcal{D}$ *denotes the range space of D; and* $\mathcal{V}$ *is the space spanned by at least q linearly independent eigenvectors.*

Proof of Theorem 3.1

To prove the theorem, define an nxn real matrix G consisting of linearly independent columns $[g_1, g_2, \ldots, g_n]$ such that

$$[g_1, g_2, \ldots, g_n] = [\alpha_1, \alpha_2, \alpha_3, \ldots, \text{Re}(\beta_1), \text{Im}(\beta_1), \ldots]$$

in any order, where α_i are the real closed-loop system eigenvectors and β_i are the complex symmetric eigenvectors. $\text{Re}(\beta_i)$ and $\text{Im}(\beta_i)$ represent the real and imaginary part of β_i, respectively. By the original definition of W, w_i represents any one of α_i, β_i or $\overline{\beta}_i$ (the complex conjugate of β_i).

For necessity suppose that $w_i \not\subset \text{Ker } c_k^T$, $i \in n$; then $(c_k^T w_i) \neq 0$, at least for one value of k for each i. This implies that $(v_i^T d_j) = 0$, $i \in n$, $j \in q$, that is $VD = 0$. This is not possible. Therefore some of the eigenvectors,

say p, must lie in Ker C. After renumbering these eigenvectors such that $c_k^T w_i = 0$, $k \in m$, $i \in p$, let the first p columns of G correspond to the real and imaginary parts of these p eigenvectors, and define $\mathcal{V}$ to be the space spanned by these p column vectors, $\{g_1, g_2, \ldots, g_p\}$. Then $\mathcal{V} \subset \text{Ker } C$. Now, for each $i \notin \{1,2,\ldots,p\}$ there exists at least one k such that $c_k^T w_i \neq 0$. This implies that $v_i^T d_j = 0$ for $j \in q$ and $i \notin \{1,2,\ldots,p\}$. Since all n columns of G span the real n-space, d_j can be written as

$$d_j = \sum_{i=1}^{n} \gamma_i g_i \qquad j \in q$$

For all $i \notin \{1,2,\ldots,p\}$, $j \in q$, $v_i^T d_j = \gamma_{ij} = 0$. Hence

$d_j = \gamma_{ij} g_1 + \gamma_{2j} g_2 + \ldots + \gamma_{pj} g_p$. Thus

$$\mathcal{D} \subset \mathcal{V} \subset \text{Ker } C \qquad (3.17)$$

From this inclusion relationship it is also easy to see that $p \geq q$, that is $\mathcal{V}$ is spanned by at least q, linearly independent eigenvectors.

With $\{g_1, g_2, \ldots, g_p\}$ defined as above, and $\mathcal{V}$ as the space spanned by these p columns vectors, the condition $\mathcal{D} \subset \mathcal{V} \subset \text{Ker } C$ implies $c_k^T w_i v_i^T d_j = 0$ $k \in m$, $i \in n$, $j \in q$. Therefore sufficiency is clear.

Remark 3.1: Note that expression (3.17) represents a weaker condition than (3.11). The result in (3.17) is a structural result rather than a synthesis result but in conjunction with the eigenvalue/eigenvector assignment technique it provides useful insight into the synthesis or the design problem.

3.6 Output Undisturbability Using Feedforward Control

It is easy to extend the necessary and sufficient condition of Wonham and Morse [12] given by expression (3.11) to include the case when feedback plus feedforward control is allowed. This can be done by simply replacing $\mathcal{D}$ in expression (3.11) by $\mathcal{L}$, where $\mathcal{L}$ denotes the range space of $L = D + BK^{FF}$, (eqn. (3.5).) The necessary and sufficient condition for disturbance localisation using feedback plus feedforward control then becomes

$$\mathcal{L} \subseteq \mathcal{V}^* \tag{3.18}$$

or, since $L = D + BK^{FF}$, expression (3.18) can also be written as

$$\mathcal{D} \subset \mathcal{V}^* + \mathcal{B} \tag{3.19}$$

Bhattacharyya [12] has derived just this result by starting with a transfer function, $G_L(s,K,K^{FF})$, relating y and

ξ, and investigating conditions under which $G_L(s,K,K^{FF}) = 0$ for all s:

$$G_L(s,K,K^{FF}) = C(sI-(A+BK))^{-1}(D+BK^{FF}) \equiv 0$$

for all s (3.20)

However, when feedforward control is used alone Bhattacharyya in [12] claims that the solvability condition is:

$$\mathcal{D} \subset \mathcal{B} \tag{3.21}$$

Unfortunately this condition is incorrect, as shown by the following discussion, (see also Shah [19]).

When feedforward control only is used the transfer function, $G_L(s,K^{FF})$, relating y and ξ is

$$G_L(s,K^{FF}) = C(sI-A)^{-1}(D+BK^{FF}) \tag{3.22}$$

The condition for complete disturbance rejection is that $y(t) = 0$ for all $t > 0$ and for every possible $\xi(t)$. This is equivalent to the requirement that

$$G_L(s,K^{FF}) \equiv 0 \quad \text{for all } s \tag{3.23}$$

Now define the subspace

$$\theta = \bigcap_{i=0}^{n-1} (\text{Ker } CA^i) \tag{3.24}$$

which is the subspace of unobservable states, that is,

(C,A) is observable if and only if $\theta = 0$. The Cayley-Hamilton Theorem insures that θ is A-invariant (e.g.: $x \in \theta$ implies $A\,x \in \theta$) (see Appendix 1). In fact θ is the largest A-invariant subspace contained in Ker C.

The solvability condition for disturbance rejection with only feedforward control allowed, is now presented in the form of a theorem.

Theorem 3.2

There exists a K^{FF} such that $G_L(s,K^{FF}) \equiv 0$ for all s if and only if

$$\mathcal{D} \subset \mathcal{B} + \theta$$

Proof of Theorem 3.2

Necessity: Bhattacharyya [12] has shown that condition (3.23) is equivalent to

$$CA^i(D+BK^{FF}) = 0 \quad \text{or } CA^iL = 0 \qquad i \in 0,1,\ldots,n-1 \tag{3.25}$$

This equivalence relationship can be shown by writing the characteristic equation of A as:

$$|G_L(s)| = s^n - a_1\,s^{n-1} - a_2 s^{n-2} \ldots\ldots - a_n = 0$$

Using a Bezoutian expansion for $(sI-A)^{-1}$ (Gantmacher [26]) it follows that

$$G_L(s,K^{FF}) = \frac{1}{|G_L(s)|} \{CB_o Ls^{n-1} + CB_1 Ls^{n-2} + \ldots$$

$$+ CB_{n-2}Ls + CB_{n-1}L\} \qquad (3.26)$$

where

$$B_o = I,\ B_j = AB_{j-1} - a_j I, \quad j \in 1,2,\ldots,n-1$$

Using (3.26) it is not difficult to show that condition (3.23) is equivalent to (3.25). From (3.25)

$$\text{Range } (D+BK^{FF}) \subset \theta \qquad (3.27)$$

Expression (3.27) implies that

$$\mathcal{D} \subset \theta + \mathcal{B} \qquad (3.28)$$

which proves necessity.

<u>Sufficiency</u>: It is possible to chose K^{FF} so that

$$\text{Range } (D+BK^{FF}) \subset \theta$$

From (3.24)

$$\text{Range } (D+BK^{FF}) \subset \theta \subset \text{Ker } C$$

hence

$$CA^{i}(D+BK^{FF}) = 0, \quad i \in 0,1,\ldots,n-1$$

which is equivalent to

$$H(s,K^{FF}) \equiv 0 \quad \text{for all } s$$

This completes proof of the theorem.

Remark 3.2: The solvability condition expressed by (3.28) is more general than the solvability condition (3.21) of Bhattacharyya [13] as shown by the following discussion. In the conventional feedforward control problem K^{FF} is determined by solving the equation:

$$D + BK^{FF} = 0 \tag{3.29}$$

Note that for completely observable linear dynamic systems, condition (3.28) reduces to condition (3.21) as presented by Bhattacharyya, and both are equivalent to eqn. (3.29). However, for the case where θ is non-zero, condition (3.21) fails but (3.28) still holds and in fact shows that the solvability condition (3.28) is more general. As is obvious from (3.28), the solvability condition for unobservable systems is weaker than that for observable systems.

In summarising the main contents of sections 3.5 and 3.6 one can say that the necessary and sufficient conditions for the existence of state feedback controllers that will decouple the effect of disturbances, $\xi(t)$, from outputs, $y(t)$, can be defined in geometric terms as follows:

1) the range space of D must lie within the maximal (A, B) - invariant subspace contained in Kernel C (Wonham and Morse [12], Bhattacharyya [13]) or that

2) the range space of D must lie within the space spanned by at least q of the closed-loop system eigenvectors (where q is the dimension of the disturbance vector, ξ), which in turn must be contained in Kernel C (Cf. Theorem 3.1).

However, in the results to be presented below these conditions are <u>not</u> tested directly. Instead it is shown via Theorems 3.3 and 3.5 that the necessary and sufficient conditions for state undisturbability can be expressed in terms of the structure of the system matrix, H, or equivalently, the structure of the closed-loop eigenvector matrix, W. Then a design procedure is presented that generates the required controller matrices based on eigenvalues and elements of the closed-loop eigenvectors chosen by the designer.

3.7 Undisturbability and Closed-loop System Structure

The following result forms the basis for design of undisturbable systems.

Theorem 3.3

*A system*characterised by the pair (H,ℓ_j) has $k(1\le k\le n)$ undisturbable state variables with respect to ξ_j, if and only if,*

i) Rank $(\ell_j) = 1$ and the pair (H,ℓ_j) is in the following form, or can be brought into this form, by a suitable reordering of the state variables:

$$\left[\begin{array}{c|c} H_1 & 0 \\ \hline H_3 & H_4 \end{array}\right], \quad \left[\begin{array}{c} 0 \\ \hline \ell_{2j} \end{array}\right] \tag{3.30}$$

where $H_1 \in R^{kxk}$, $H_4 \in R^{(n-k)x(n-k)}$, $\ell_{2j} \in R^{(n-k)}$

and $x_1, x_2, \ldots, x_k$ are the k undisturbable state variables of the pair (H,ℓ_j).

or

ii) $\ell_j = 0$. This corresponds to a trivial case.

To prove the theorem the following lemma will be used, Herstein [27].

---

* Note: H_1, H_3, H_4 and ℓ_{2j} can be time varying since the only restriction is $H_2 = f(A_2,B_1,K_2) = 0$.

Lemma 3.1

Let $\mathcal{M}$ be a nonempty set of nxn matrices in $\mathcal{R}^n$; if $\mathcal{W}(\mathcal{W}\subset\mathcal{R}^n)$ is an invariant subspace under $M \in \mathcal{M}$ and is of dimension (n-k) over $\mathcal{R}$, then there exists a basis of $\mathcal{R}^n$ over $\mathcal{R}$ such that every $M \in \mathcal{M}$ in this basis is of the form,

$$\left[\begin{array}{c|c} M_1 & 0 \\ \hline M_3 & M_4 \end{array}\right] \tag{3.31}$$

where $M_1 \in \mathcal{R}^{k\mathrm{x}k}$ and $M_4 \in \mathcal{R}^{(n-k)\mathrm{x}(n-k)}$. Furthermore, in this basis M_4 is the matrix representation of the restriction of M to $\mathcal{W}$ and M_1 is the restriction of M to $\mathcal{R}^n/\mathcal{W}$. The proof of the lemma is straightforward and is given in Appendix A.

Proof of Theorem 3.3

Let the first k state variables of the system characterised by the pair (H,ℓ_j) be represented by xl, i.e.

$$xl(t) = [x_1(t)\ x_2(t)\ \ldots\ x_k(t)]^T$$

Sufficiency is clear from the structure of the pair (H,ℓ_j) in eqn. (3.30), since

$$xl^{\xi}(t) = 0 \text{ for all } t > 0 \text{ and arbitrary } \xi_j(t).$$

<u>Necessity</u> Consider a system characterised by the pair (H, ℓ_j) which has k state variables which are undisturbable with respect to ξ_j. Let the system be brought into the form of eqn. (3.32), by reordering the state variables if necessary, such that $x = [x1\ x2]^T$ where $x1 = [x_1, x_2, \ldots, x_k]^T$ contains the k undisturbable state variables and $H_1 \in R^{k \times k}$, $H_4 \in R^{(n-k)\times(n-k)}$ and $\ell_{1j} \in R^k$:

$$\begin{bmatrix} \dot{x}1^{\xi}(t) \\ \hline \dot{x}2^{\xi}(t) \end{bmatrix} = \left[\begin{array}{c|c} H_1 & H_2 \\ \hline H_3 & H_4 \end{array}\right] \begin{bmatrix} x1^{\xi}(t) \\ \hline x2^{\xi}(t) \end{bmatrix} + \begin{bmatrix} \ell_{1j} \\ \hline \ell_{2j} \end{bmatrix} \xi_j(t) \tag{3.32}$$

Since undisturbability implies $x1^{\xi}(t) = 0$ for arbitrary $\xi_j(t)$ and for all $t > 0$, eqn. (3.32) reduces to the following two equations:

$$0 = H_2 x2^{\xi}(t) + \ell_{1j}\xi_j(t) \tag{3.33}$$

$$\dot{x}2^{\xi}(t) = H_4 x2^{\xi}(t) + \ell_{2j}\xi_j(t) \tag{3.34}$$

Let $\mathcal{W}$ be the H-cyclic subspace generated by ℓ_j, as defined below.

$$\mathcal{W} \equiv \ell_j + H\ell_j + H^2\ell_J + \ldots + H^{n-1}\ell_j \tag{3.35}$$

In fact $\mathcal{W}$ is the H-invariant subspace of states which ξ_j can control. Let $\dim(\mathcal{W}) = p$. Then it is clear that

$p \leq (n-k)$. Since the effect of ξ_j on x is given by eqn. (3.34), it is correct to say that the matrix representation of the restriction of H to $\mathcal{W}$ in the standard basis is H_4. Two cases will now be considered: (i) dim $(\mathcal{W}) = p = n-k$ and (ii) $\dim(\mathcal{W}) = p < (n-k)$.

<u>Case 1</u>: $p = n-k$

Since the matrix representation of the restriction of H to $\mathcal{W}$ in the standard basis is H_4, then by Lemma 1 the p standard basis vectors: $\{e_{k+1}, e_{k+2}, \ldots e_n\}$ span the H-invariant subspace $\mathcal{W}$. This implies that $\ell_{1j} = 0$, and since x2 is disturbable, it follows from eqn. (3.33) that $H_2 = 0$.

<u>Case 2</u>: $p < (n-k)$

Now H_4 can no longer be the matrix representation of the restriction of H to $\mathcal{W}$, but it still follows that p of the (n-k) standard basis vectors, $\{e_{k+1}, e_{k+2}, \ldots, e_n\}$, span $\mathcal{W}$. This once again implies that $\ell_{1j} = 0$, and as in the previous case, that $H_2 = 0$.

In either case $\ell_{1j} = 0$ and $H_2 = 0$ which means that the undisturbable system (3.32) exists in, or can be brought into, the form of eqn. (3.30). This shows necessity and completes the proof of the theorem.

Remark 3.3: The condition rank $(\ell_j) = 1$ is equivalent to the condition $k < n$ in the sense that each implies the other. The significance of the condition, rank $(\ell_j) = 1$ (or $k < n$), is that a system characterised by the pair (H,ℓ_j) has at least one disturbable state variable. This conclusion also follows directly from geometric concepts and the principle of conservation of energy.

Remark 3.4: A non-trivial system (i.e. $\ell_j \neq 0$) characterised by the pair (H,ℓ_j) has no undisturbable state variables (i.e. it is completely state disturbable) if either ℓ_j or H do not contain any zero elements.

Theorem 3.4

The i^{th} output, y_i, of a system characterised by (H,ℓ_j,c_i^T) is undisturbable with respect to $\xi_j(t)$, if and only if

$$W \subset Ker\ c_i^T \tag{3.36}$$

Proof of Theorem 3.4

For y_i to be undisturbable with respect to ξ_j then $y_i^{\xi}(t) = 0$ for all t and every possible ξ_j (t), or equivalently that

$$c_i[sI-H]^{-1}\ell_j \equiv 0 \text{ for all s.}$$

Using eqn. (3.26) the above condition can be shown to be equivalent to

$$c_i H^m \ell_j = 0 \qquad m \in 0,1,\ldots,n-1 \qquad (3.37)$$

Both sufficiency and necessity then follow easily from eqn. (3.37) and eqn. (3.35) which defines $\mathcal{W}$.

Remark 3.5 Note that Theorem 3.3 can also be derived as a direct consiquence of Theorem 3.4, in the following manner. For system (3.32), where $x_1, x_2, \ldots, x_k$ are the undisturbable state variables, by Theorem 3.4 the following necessary and sufficient condition needs to be satisfied:

$$\mathcal{W} \subset \text{Ker } [e_1, e_2, \ldots, e_k]^T \qquad (3.38)$$

where $\mathcal{W}$ is defined by eqn. (3.35) and $e_1, e_2, \ldots, e_k$ are the standard basis vectors. This clearly means that $\{e_{k+1}, e_{k+2}, \ldots, e_n\}$ span $\mathcal{W}$. Hence it follows that $\ell_{1j} = 0$, and from eqn. (3.33), that $H_2 = 0$.

Eqn. (3.38) also provides an interpretation of $\mathcal{V}^*$ for the case when $y_i = x_i$, i=1, ..., k. For this case, $\mathcal{V}^*$ is the largest subspace of states which can be made unobservable at y = Cx, where m = k.

3.8 Closed-loop Eigenvector Structure and Undisturbability

The following theorem relates the property of undisturbability to the closed-loop system eigenvectors.

Theorem 3.5:

Consider a system characterised by the pair (H,ℓ_j) where H has distinct eigenvales; this system has $k(1\leq k<n)$ undisturbable state variables with respect to disturbance, ξ_j, if and only if:

i) the closed-loop system eigenvector matrix, W, is of the following form or can be brought into the following form through rearrangement of its columns:

$$W = \left[\begin{array}{c|c} W_1 & 0 \\ \hline W_3 & W_4 \end{array}\right] \qquad (3.39)$$

where $W_2 \in \mathbb{C}^{kxk}$, $W_4 \in \mathbb{C}^{(n-k)x(n-k)}$, *and*

ii) ℓ_j is of the form:

$$\ell_j = \left[\begin{array}{c} 0 \\ \hline \ell_{2j} \end{array}\right] \qquad (3.40)$$

where $\ell_{2j} \in R^{(n-k)}$ *and the first* k *elements are zero and correspond to the* k *undisturbable state variables.*

To prove the theorem the following Lemma will be used.

Lemma 3.2

Let $\mathcal{M}$ be a set of nxn matrices in R^n with distinct eigenvalues. Then $M \in \mathcal{M}$ is of the following form:

$$M = \left[\begin{array}{c|c} M_1 & 0 \\ \hline M_3 & M_4 \end{array}\right] \qquad (3.41)$$

where $M_1 \in R^{kxk}$, $M_4 \in R^{(n-k)x(n-k)}$, and $1 \leq k < n$, if and only if W, the eigenvector matrix of M, is in the form shown in eqn. (3.39).

Proof of Lemma 3.2

With W in the form of eqn. (3.39), sufficiency is clear since $M = W\Lambda V$, where $V = W^{-1}$ and Λ denotes the nxn diagonal matrix whose diagonal elements are the eigenvalues of M.

To prove necessity, note that since M in eqn. (3.41)

is a lower quasi-triangular matrix, its determinant is the product of the determinants of the diagonal blocks, Gantmacher [26], i.e. $|M| = |M_1|\,|M_4|$. This means that,

$$\sigma(M) = \sigma(M_1) \cup \sigma(M_4)$$

where $\sigma(M)$ denotes the spectrum of M. For this part of the proof assume that Λ and W are partitioned so as to be compatible with the partitions of M in eqn. (3.41):

$$\Lambda = \left[\begin{array}{c|c} \Lambda_1 & 0 \\ \hline 0 & \Lambda_2 \end{array}\right] \qquad W = \left[\begin{array}{c|c} W_1 & W_2 \\ \hline W_3 & W_4 \end{array}\right] \tag{3.43}$$

where $\Lambda_1 \in \mathbb{C}^{k\mathrm{x}k}$, $\Lambda_2 \in \mathbb{C}^{(n-k)\mathrm{x}(n-k)}$ and they are arranged such that:

$$\sigma(M_1) = \text{diagonal elements of } \Lambda_1 \tag{3.44}$$

$$\sigma(M_4) = \text{diagonal elements of } \Lambda_2 \tag{3.45}$$

Since W is the eigenvector matrix of M, it follows that:

$$MW = W\Lambda \tag{3.46}$$

From eqns. (3.41), (3.43) and (3.46), it follows that:

$$M_1W_1 = W_1\Lambda_1 \tag{3.47}$$

and

$$M_1W_2 = W_2\Lambda_2 \tag{3.48}$$

Clearly, W_1 in eqn. (3.47) is the eigenvector matrix for M_1. Since $\Lambda_1 \neq \Lambda_2$, eqn. (3.48) implies that $W_2 = 0$, which proves necessity.

<u>Proof of Theorem 3.5</u>

In view of the necessary and sufficient conditions for undisturbability in Theorem 3.3, it suffices to prove that the closed-loop system matrix H is of the form

$$H = \left[\begin{array}{c|c} H_1 & 0 \\ \hline H_3 & H_4 \end{array}\right] \tag{3.49}$$

where $H_1 \in R^{k\times k}$ and $H_4 \in R^{(n-k)\times(n-k)}$, if and only if its eigenvector matrix W is, or by rearrangement of its columns, can be brought into the form shown in eqn. (3.39). Clearly, the proof for sufficiency and necessity both follow from the result of Lemma 3.2.

<u>Remark 3.6</u> The assumption that H has distinct eigenvalues entails little loss of generality, since almost any feedback matrix, K, will yield a closed-loop system with distinct eigenvalues (Davison and Wang [28].

<u>Remark 3.7</u>: The extension of the results of Theorem 3.5 to discrete-time systems is obvious.

Theorem 3.5 helps to relate the two sets of necessary and sufficient conditions considered in Theorems 3.1 and 3.3 in the following way. The necessary and sufficient conditions for disturbance localisation in Theorem 3.1 are expressed in terms of the subspace spanned by some of the closed-loop eigenvectors, i.e. the (A,B)-invariant subspace $\mathcal{V}$. The necessary and sufficient conditions for undisturbability in Theorem 3.3 are given in terms of the structure of coefficient matrices of the state-space model. Theorem 3.5 is basically an extension of Theorem 3.3 and shows that the two sets of necessary and sufficient conditions in Theorems 3.1 and 3.3 are equivalent. In contrast to previous results on disturbance localisation (Wonham and Morse [12], Bhattacharyya [13]), the results in Theorems 3.3 and 3.5 are structural results. However, in conjunction with the eigenvalue/eigenvector assignment technique these results lead to a very simple design procedure.* In contrast to that the results of Wonham and Morse [12] do not lend themselves easily to computer implementation. One of the main difficulties is the evaluation of $\mathcal{V}^*$ which is subject to the hazards of ill-conditioning, Bhattacharyya et al. [23].

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* The design procedure based on Theorem 3.5 allows the user to specify the elements of W_1, which is not possible in previous methods [12].

3.9 Output Undisturbability and Closed-loop System Structure

When the system outputs are a subset of the state variables then the condtions for state undisturbability (Theorems 3.3 and 3.5) can be applied directly to determine output undisturbability. However, when outputs are a linear combination of the state variables then conditions for state undisturbability do not apply directly. In this case output undisturbability can be handled by transforming the system prior to applying the conditions of Theorem 3.3. The first step is to define a new vector, $\underline{z}$, of dimension n, such that (cf.[36]):

$$z = \begin{bmatrix} C \\ --- \\ E \end{bmatrix} x \equiv \underset{\sim}{Q}^{-1} x \qquad (3.50)$$

where C is the mxn output matrix of the original system (assumed to be of full rank) and E is an (n-m)xn matrix chosen arbitrarily such that its (n-m) rows are linearly independent of the rows of C. Under these conditions Q exists, and the original system defined by eqn. (3.1) becomes:

$$\dot{z} = Q^{-1} A\ Q\ z + Q^{-1} B\ u + Q^{-1} D\ \xi \qquad (3.51)$$

$$y = C\ Q\ z$$

The closed-loop systems can now be represented by eqn. (3.3) where:

$$H = Q^{-1}A\ Q + Q^{-1}B\ K \quad \text{and} \tag{3.52}$$

$$L = Q^{-1}D + Q^{-1}B\ K^{FF}$$

The conditions for state undisturbability can therefore be applied directly to the transformed system and since $\{z_i = y_i, i=1,\ldots,m\}$ they provide a basis for the design of systems with undisturbable outputs. This is illustrated in one of the following examples (Section 3.12).

3.10 Discussion of Theoretical Results

3.10.1 The relationship between undisturbability and uncontrollability

From a geometric viewpoint the essential difference between undisturbability and uncontrollability is the following. With $\mathcal{W}$ defined by (3.35), then from Lemma 3.1 it is clear that an open-loop system characterised by the pair (A,d_j) is "uncontrollable" if and only if a linear transformation can bring it into the form of (3.30). On the other hand, the pair (A,d_j) has k undisturbable state variables if and only if it can be brought into the form of (3.30) through a particular class of

of linear transformations, namely, permutations of coordinates to reorder the state variables. This class of linear transformations also allows the undisturbable state variables to be identified. These considerations lead to the following conclusions:

i) complete state controllability of the pair (A,d_j) implies complete state disturbability of the pair (A,d_j). However, the converse is not always true.

ii) if some of the state variables of the system (A,d_j) are undisturbable, then it follows that the pair (A,d_j) is uncontrollable. However, the converse is not necessarily true. Even if the pair (A,d_j) is uncontrollable it is possible that (A,d_j) is completely state disturbable. Note that analogous results also apply for output disturbability.

As is evident from the above discussion, undisturbability of the pair (A,d_j) cannot always be determined from controllability concepts. However, uncontrollability of the pair (A,d_j) can often be inferred by application of the undisturbability criterion; that is, the pair (A,d_j) is uncontrollable if the pair (A,d_j) has some undisturbable state variables (cf. Section 3.10.2). These relationships are graphically summarised in the venn diagram of Figures 3.2 and 3.3.

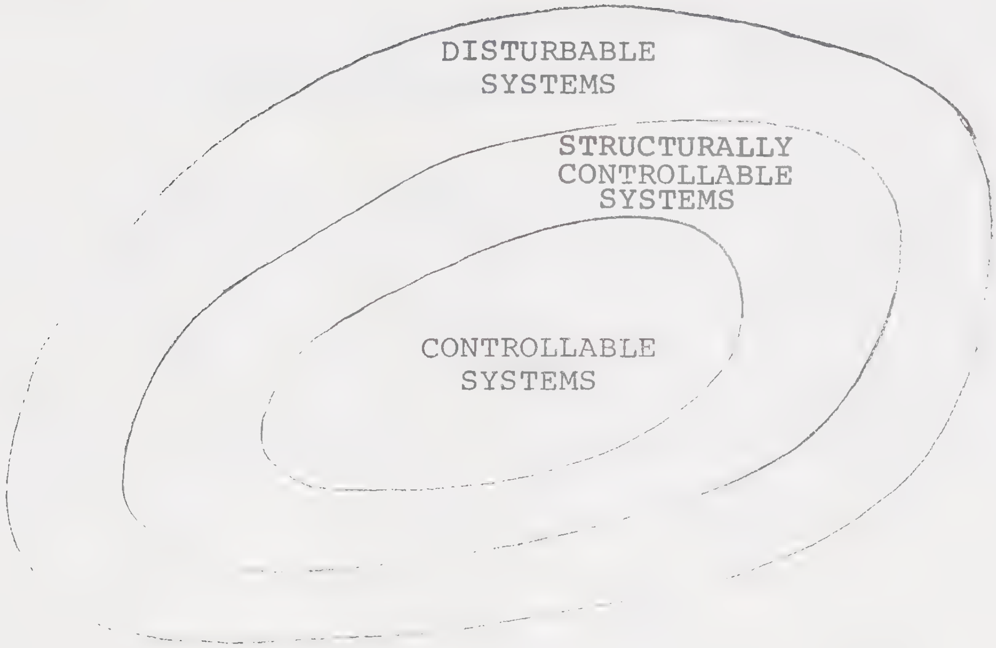

Figure 3.2: Venn diagram to graphically illustrate the relationship between disturbable, controllable and structurally controllable systems.

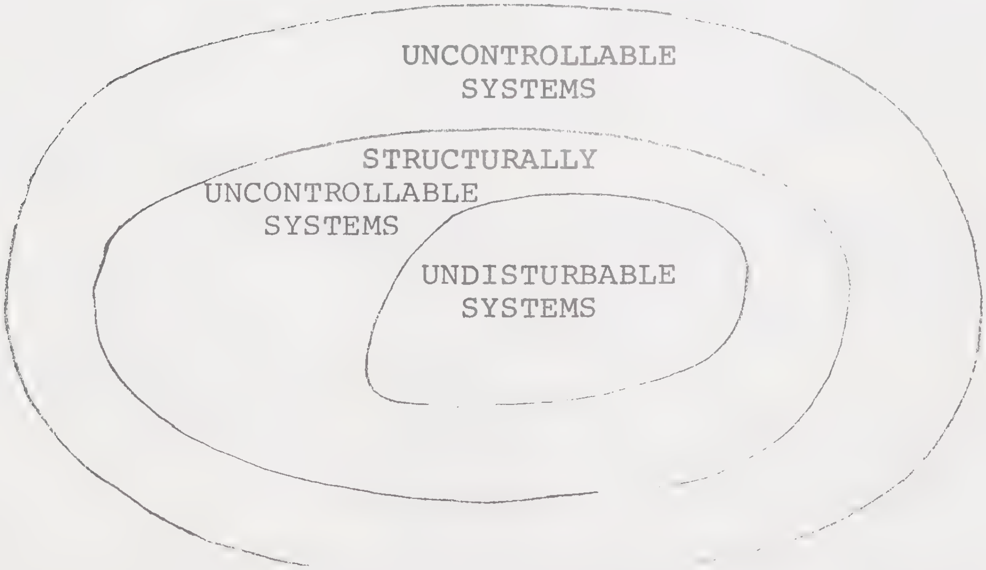

Figure 3.3: Venn diagram to graphically illustrate the relationship between undisturbable, uncontrollable and structurally uncontrollable systems.

3.10.2 The relationship between undisturbability and structural controllability

The class of undisturbable systems are a special class of uncontrollable systems, namely those which are not "structurally controllable", a concept recently introduced by Lin [29]. Structural controllability is a property that is useful in determining the controllability characteristics of a system from a knowledge of only the zero entries in matrices of the pair (A,B). A system which is not structurally controllable can be brought into two possible forms through permutation of coordinates. One of these forms is identical to the form of (3.30). Thus there is a close relationship between undisturbability and structural uncontrollability. Undisturbability can also be investigated without knowledge of the exact numerical values of the non-zero elements of matrices A and D since only knowledge of the zero elements of these matrices is required.

Theorem 3.3, which is based on algebraic concepts, can be used to show the equivalence of the following two properties for the system in eqn. (3.1).

i) The signal flowgraph of the nontrivial pair (A,d_j) does not contain any non-accessible state variables, (i.e. the system is completely disturbable).

ii) there is no permutation of coordinates which brings the pair (A,d_j) into the form of (3.30).

By contrast, Lin [29] has used a graph theoretic approach to study structural controllability and to show the equivalence of results similar to (i) and (ii) above for structural controllability. The relationship between disturbability, controllability and structural controllability is summarised in the venn diagram of Figures 3.2 and 3.3.

3.10.3 The relationship between undisturbability and zeros of Rosenbrock's system matrix.

In Section 3.10.1 the relationship between undisturbability and uncontrollability was considered. Earlier we also remarked about the relationship between undisturbability and unobservability. In the ensuing discussion we will investigate a more definitive mathematical relationship between these three concepts by considering Rosenbrock's systems matrix. In Rosenbrock [35] the ideas of controllability and observability have been embodied into the following definitions.

Controllability: The pair (A,B) is controllable if and only if (sI-A) and B are relatively left prime.

Observability: The pair (C,A) is observable if and only if (sI-A) and C are relatively right prime.

According to Rosenbrock [35], the eigenvalue corresponding to a mode which is completely decoupled from an input is said to be an input decoupling (i.d.) zero. In the same manner, the eigenvalue corresponding to a mode which is completely decoupled from an output is said to be an output decoupling (o.d.) zero. Finally, the eigenvalue corresponding to a mode which is completely decoupled both from the input as well as the output is said to be an input output decoupling (i.o.d.) zero. With this brief review of basic system matrix terminology we now present the following result in the form of a theorem:

<u>Theorem 3.6</u>

An n^{th} order system characterised by the triple (H,L,C) is completely output undisturbable if and only if

$$z_i + z_o - z_{io} = n$$

where z_i *denotes the number of i.d. zeros,* z_o *denotes the number of o.d. zeros, and* z_{io} *is the number of i.o.d. zeros.*

Remark 3.8: The statement of Theorem 3.6 in effect says that (sI-H) and L are relatively (left) prime and (sI-H) and C are relatively (right) prime. Note also that the triple (A,D,C) can be regarded as a special case of (H,L,C) and hence the result of Theorem 3.6 applies to both open as well as closed-loop systems.

Proof of Theorem 3.6:

Sufficiency is clear since $z_i + z_o - z_{io} = n$ implies that the minimal system order = 0, that is to say that $y(s) = C(sI-H)^{-1}L \equiv 0$.

Necessity: For this assume that the outputs are a subset of the state variables. If this is not the case, the system can always be transformed into a set of new coordinates where the outputs are a subset of the new state variables (cf. Section 3.9). Once this is done then using the arguments of Theorems 3.3 and 3.4 it can be shown that the system matrix P(s) for the undisturbable system characterised by the triple (H,L,C) is of the form:

$$P(s) = \left[\begin{array}{cc|c} sI-H_1 & 0 & 0 \\ -H_3 & sI-H_4 & L_2 \\ \hline -I_m & 0 & 0 \end{array}\right]$$

With the above form of P(s), the transfer function between y(s) and ξ(s) can be evaluated as:

$$y(s) = C(sI-H)^{-1}L, \quad \text{or}$$

$$y(s) = [I_m\ 0]\begin{bmatrix} I & 0 \\ 0 & sI-H_4 \end{bmatrix}\begin{bmatrix} sI-H_1 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -H_3 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & sI-A_4 \end{bmatrix}^{-1} x$$

$$x \begin{bmatrix} sI-H_1 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} = [I_m\ 0]\begin{bmatrix} I & 0 \\ -H_3 & I \end{bmatrix}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} \equiv 0$$

From the above matrix equality it is clear that the i.d. zeros are the poles of $|sI-H_1|$, i.e. the eigenvalues of H_1, and the o.d. zeros are the poles of $|sI-H_4|$ or the eigenvalues of H_4. Now H is a quasi-triangular matrix therefore eigenvalues of H are $\sigma(H_1) \cup \sigma(H_4)$. Since H is n^{th} order, $z_i + z_o = n$. Note that in some cases the {set of i.d. zeros}$\cap${set of o.d. zeros} $\neq 0$ (e.g. if L_2 has one or more zero rows), in this case the system has (i.o.d.) zeros as well. Because of this $z_i+z_o > n$ and hence we need to subtract z_{io} to satisfy the equality: $z_i + z_o - z_{io} = n$. This completes the proof of the theorem.

Remark 3.9: It is interesting to note that conceptually the proof of this theorem becomes readily apparent when the definition of undisturbability is considered from a signal flowgraph terminology - just as the concept of decoupling zeros becomes clear from a signal flowgraph of a system in the Kalman canonical form. For example the eigenvalues associated with the non-accessible modes or the undisturbable modes can be identified with i.d. zeros and the eigenvalues associated with the accessible modes (the disturbable modes) can be identified with the o.d. zeros since there is no path from ξ to y.

3.11 Design Considerations

It was mentioned in Section 3.5 that the necessary and sufficient conditions for disturbance localisation reported by Wonham and Morse [12] and Bhattacharyya [13] were expressed in terms of $\mathcal{V}^*$, the maximal (A,B)-invariant subspace contained in Kernel C. Athough $\mathcal{V}^*$ plays a key role in a number of control problems (Morse and Wonham, [30]) it is not easily calculated (Bhattacharyya [31], Bhattacharyya et al. [23]). By contrast, the result of Theorem 3.5 in conjunction with eigenvalue and eigen-vector assignment techniques allows insight into the design problem and is also useful from a synthesis point

In this section a design procedure to achieve disturbance localisation will be developed based on Theorem 3.5 and the eigenvalue/eigenvector assignment technique (Srinathkumar and Rhoten [32] (cf. Chapter 2)). From Theorem 3.5 it is clear that in order to achieve disturbance localisation, K must be specified so that the closed-loop eigenvectors are in the form of eqn. (3.39) and if feedforward control is necessary, then K^{FF} must be specified so that L is in the form of eqn. (3.40). The manner in which the design objectives of arbitrary eigenvalue assignment and disturbance localisation influence the design procedure will now be considered.

The closed-loop system matrix can be expressed as:

$$H = A + BK = W\Lambda V \tag{3.53}$$

where V is defined by, $V = W^{-1}$. Eqn. (3.53) can be partitioned as:

$$\left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] + \begin{bmatrix} B_1 \\ \\ B_2 \end{bmatrix} [K_1 \quad K_2] =$$

$$\left[\begin{array}{c|c} W_{11} & W_{12} \\ \hline W_{21} & W_{22} \end{array}\right] \left[\begin{array}{c|c} \Lambda_1 & 0 \\ \hline 0 & \Lambda_2 \end{array}\right] \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{21} & V_{22} \end{array}\right] \tag{3.54}$$

In equation (3.54) the partitions of H and W are denoted by double subscripts (e.g. W_{11}) to avoid confusion with the partitions in eqns. (3.30) and (3.29) which have different dimensions. Matrices H_{11}, A_{11}, B_1, K_1, W_{11}, Λ_1, and V_{11} are rxr matrices where r is the number of inputs; H_{22}, A_{22}, W_{22}, Λ_2, and V_{22} are (n-r)x(n-r) matrices and B_1 is assumed to be nonsingular. (If necessary the state variables can be reordered to make B_1 nonsingular. Note that W and consequently Λ and V can be complex matrices.) From eqn.(3.54) it follows that Range $(B_1) = R^r$, and thus a K exists such that the rxr elements of W_{11} and the rx(n-r) elements of W_{12} can be <u>chosen arbitrarily</u>, subject only to the requirement that W be nonsingular (since $V = W^{-1}$). Once Λ_1 and Λ_2 (i.e. all the closed-loop system eigenvalues) and W_{11} and W_{12} are chosen, W_{21} and W_{22} can be calculated (Srinathkumar and Rhoten, [32]) from eqns. (3.55) and (3.56) which follow from eqn. (3.54):

$$W_{21}\Lambda_1 - PW_{21} = TW_{11} + SW_{11}\Lambda_1 \tag{3.55}$$

$$W_{22}\Lambda_2 - PW_{22} = TW_{12} + SW_{12}\Lambda_2 \tag{3.56}$$

where

$$S = B_2 B_1^{-1}$$

$$T = A_{21} - SA_{11} \tag{3.57}$$

$$P = A_{22} - SA_{12}$$

Then K can be computed from eqn. (3.53) by using the pseudoinverse of B (Chapter 2, see also Shah et al. [33]). Note that the pseudoinverse solution for K provides an exact solution to eqn. (3.53) since W_{21} and W_{22} have been chosen so as to satisfy eqns. (3.55) and (3.56).

Specifying the first r elements of each of the n eigenvectors requires only (rxn-n) degrees of freedom, i.e. (rxn-n) elements of K, providing that at least one of the r specified values in each eigenvector is nonzero. The remaining n degrees of freedom can be used to arbitrarily assign the closed-loop eigenvalues. Thus in general, simultaneous eigenvalue and eigenvector assignment are possible provided that at least one of the first r entries in each eigenvector is chosen to be nonzero, and that the final eigenvector matrix W is nonsingular.

3.11.1 Steps in the design procedure

The main steps in the design procedure can be summarized as follows:

i) After formulating the control objectives, matrix D is inspected to determine if the selected state or output variables can be made undisturbable by feedback control or whether feedback-feedforward control is required. If D has the form in eqn. (3.40), then only feedback control is required; if not, then feedforward-feedback control must be used.

ii) Matrices W_{11} and W_{12} in eqn. (3.54) are assigned according to the result in Theorem 3.5.

iii) The closed-loop eigenvalues in Λ_1 and Λ_2 are assigned.

iv) Matrices W_{21} and W_{22} are calculated from eqns. (3.55) and (3.56) and a check is made to determine whether the resulting eigenvector matrix, W is nonsingular. If W is singular, step ii) is repeated with a different choice of W_{11} and W_{12}.

v) Feedback matrix K is calculated from eqn. (3.53) using the pseudoinverse of B.

vi) If feedforward control is used, K^{FF} is then designed so that the appropriate elements in L are zero. (cf. Section 3.12 for illustrative examples).

Note that the present design procedure suffers from the disadvantage that if B_1 is singular, then not all of the control objectives can be met and therefore new control objectives have to be formulated such that the final B_1 obtained after reordering state variables is nonsingular. However, there are exceptions to this case; that is in some instances the original design objectives can be met even if B_1 is singular, as for example, the distillation column example considered in Chapter 4. Moore [34] has recently proposed a method for eigenvalue/eigenvector assignment which does not require B_1 to be nonsingular and thus provides an important alternative to the approach of Srinathkumar and Rhoten [32] as considered in Chapter 2.

3.11.2 Simultaneous disturbance localisation and eigenvalue assignment

From a design point of view, it is desirable to be able to arbitrarily assign all n closed-loop eigenvalues as well as achieve disturbance localisation. However, it is not always possible to satisfy both objectives if the eigenvector/eigenvalue design technique presented in Chapter 2 is used (see also Srinathkumar and Rhoten [32] and Shah et al. [33]). In particular, when the desired number of undisturbable state variables is less than the

number of control variables (i.e. $k<r$), then simultaneous eigenvalue assignment and disturbance localisation is possible, but when $k=r$ not all eigenvalues can be arbitrarily assigned. The reason for this limitation will be considered in the following discussion.

In order to make k state variables $x_1, \ldots, x_k$ undisturbable, the $k\times(n-k)$ matrix, W_2, must satisfy the condition in Theorem 3.5 that $W_2 = 0$. However, in the eigenvalue/eigenvector design algorithm, arbitrary eigenvalue assignment is possible only if there is at least one nonzero element among the first r elements of each closed-loop eigenvector. These two conditions imply that if $k<r$, simultaneous disturbance localisation and arbitrary eigenvalue assignment can be achieved by setting $W_2 = 0$ and assigning nonzero values to at least one of the remaining r-k elements in each column of W_{12} (cf. eqn. (3.54)). By contrast, when $k=r$ some closed-loop eigenvalues cannot be arbitrarily assigned. In this case Theorem 3.5 requires that $W_2 = W_{12} = 0$ for disturbance localisation but when $W_{12} = 0$, it follows from eqn. (3.56) that the n-r eigenvalues of Λ_2 are the eigenvalues of P and thus they cannot be arbitrarily assigned. Consequently, when $k=r$ the stability of the closed-loop system depends on the eigenvalues of P. Necessary and sufficient conditions for simultaneous disturbance localisation and

closed-loop stability have recently been reported by Fabian and Wonham [17] and Chang and Rhodes [18].

3.12 Illustrative Examples

To illustrate the concepts and design procedures of Section 3.11 two examples will be considered.

Example 3.1:

A hypothetical third order system with 2 inputs, 2 outputs and 2 disturbances is shown in eqns. (3.58) and (3.59):

$$A = \begin{bmatrix} -1.25 & 0.75 & -0.75 \\ 1.0 & -1.5 & -0.75 \\ 1.0 & -1.0 & -1.25 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \tag{3.58}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ 4 & 2 \\ 0 & 2 \end{bmatrix} \tag{3.59}$$

The eigenvalues of A are -0.5, -1.25 and -2.25. Theorem 3.3 and inspection of the system in eqns. (3.58) and (3.59) reveal that all of the state and output variables are disturbable with respect to each disturbance, since A contains no zero elements.

Suppose that the primary control objective is to make x_1 undisturbable with respect to disturbance, ξ_1. For convenience, let h_{ij} and w_{ij} denote the $(i,j)^{th}$ elements of H and W respectively. Then k=1 and from Theorem 3.5 it follows that $W_2 = [w_{12}\ w_{13}]$ must be chosen so that $w_{12} = w_{13} = 0$. This choice means that $h_{12} = h_{13} = 0$ and thus H and d_1, the first column of D, are in the required form of eqn. (3.30). The eigenvalue/eigenvector assignment method of Section 3.11 can be used to calculate the remaining elements of W. Since in this example, r = 2, all six elements in the first two rows of W (i.e. W_{11} and W_{12} in eqn. (3.54) can be arbitrarily assigned. For example, W_{11} and W_{12} can be specified as:

$$W_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad W_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.60}$$

Since $W_{12} \neq 0$ all three closed-loop eigenvalues can be assigned arbitrarily. If the closed-loop eigenvalues are selected as -3, -4 and -5, then

$$\Lambda_1 = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}, \qquad \Lambda_2 = [-3] \tag{3.61}$$

With W_{11}, W_{12}, Λ_1 and Λ_2 specified, W_{21} and W_{22} can be evaluated from eqns. (3.55) and (3.56) and K calculated from eqn. (3.54) to give:

$$W = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 1 & 1 & 1 \\ 0 & -0.100 & -0.357 \end{array}\right] \qquad K = \begin{bmatrix} -1.594 & -1.906 & 2.938 \\ 0.438 & 3.062 & -5.125 \end{bmatrix}$$

The closed-loop response to a unit step change in ξ_1 is shown in Fig. 3.4 and illustrates that x_1 has been made undisturbable with respect to ξ_1. It is interesting to note that x_3 is only slightly affected by ξ_1. This is due to the presence of a small (but nonzero) element in h_{32}.

Example 3.2:

Again consider the system in eqns. (3.58) and (3.59) but now assume that the control objective is to make x_1 and x_3 undisturbable with respect to both ξ_1 and ξ_2. Clearly, this is a much more ambitious design objective than the one considered in Example 3.1.

In view of the structure of the matrices in Theorems 3.3 and 3.5 it is convenient to reorder the state vector so that state variables, x_1 and x_3, which are to be made undisturbable appear as the first two state variables. Thus the new state vector, $\tilde{x}$, is defined by:

$$\tilde{x}^T = [\tilde{x}_1 \ \tilde{x}_2 \ \tilde{x}_3] = [x_1 \ x_3 \ x_2] \qquad (3.63)$$

Similarly, let $\tilde{A}$ and $\tilde{D}$ denote the appropriate permutations of matrices A and D corresponding to the state vector $\tilde{x}$,

$$\tilde{A} = \left[\begin{array}{cc|c} -1.25 & -0.75 & 0.75 \\ 1.0 & -1.25 & -1.0 \\ \hline 1.0 & -0.75 & -1.5 \end{array}\right], \qquad \tilde{D} = \left[\begin{array}{cc} 0 & 1 \\ 0 & 2 \\ \hline 4 & 2 \end{array}\right] \tag{3.64}$$

and let $\tilde{H}$ denote the corresponding permutation of H. Theorem 3.3 indicates that in order for $\tilde{x}_1$ and $\tilde{x}_2$ to be undisturbable with respect to ξ_1 and ξ_2, it is necessary that $\tilde{h}_{13} = \tilde{h}_{23} = 0$ and the first two rows in $\tilde{L}$ must also be zero. Since $\tilde{d}_{11} = \tilde{d}_{21} = 0$, it is possible to design K so that $\tilde{x}_1$ and $\tilde{x}_2$ are undisturbable with respect to ξ_1. However, since $\tilde{d}_{12}$ and $\tilde{d}_{22}$ are nonzero, $\tilde{x}_1$ and $\tilde{x}_2$ cannot be made undisturbable with respect to ξ_2 by using only state feedback control. In this case feedforward control of ξ_2 must be used in conjunction with feedback control.

Consider the feedback-feedforward control law of eqn. (3.65):

$$u(t) = Kx(t) + K^{FF}\ \xi_2(t) \tag{3.65}$$

Combining eqns. (3.65) and (3.1) gives the closed-loop system

$$\dot{x} = H^{x}(t) + L\xi(t) \tag{3.66}$$

The corresponding closed-loop system expressed in terms of $\tilde{x}$ is:

$$\dot{\tilde{x}} = \tilde{H}\tilde{x}(t) + \tilde{L}\xi(t) \tag{3.67}$$

where $\tilde{H} = \tilde{A} + \tilde{B}\tilde{K}$, $\tilde{L} = \tilde{B}\tilde{K}^{FF} + \tilde{D}$.

In order for $\tilde{x}_1$ and $\tilde{x}_2$ to be undisturbable with respect to ξ_2, the second column of $\tilde{L}$, $\tilde{\ell}_2$, must have the form $\tilde{\ell}_2^T = [0,0,\alpha]$ where α is to be determined. Then K^{FF} and α can be calculated from

$$\tilde{B}\, K^{FF} + \tilde{d}_2 = \tilde{\ell}_2 \tag{3.68}$$

to give

$$K^{FF} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \text{and} \quad \alpha = 3.0 \tag{3.69}$$

To complete the controller design, the state feedback matrix K must be calculated. The undisturbability objectives and Theorem 3.5 require that $\tilde{W}_2$ must be chosen as $\tilde{W}_2 = \tilde{W}_{12} = 0$ while $\tilde{W}_{11}$ can be selected arbitrarily as

$$\tilde{W}_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{3.70}$$

Since $\tilde{W}_{12} = 0$ and k=r, it is not possible to arbitrarily assign all of the closed-loop eigenvalues, as discussed in Section 3.11. In fact eqn. (3.57) indicates that P is a scalar quantity with a numerical value of -4.25 and consequently, one of the closed-loop eigenvalues, λ_3, has a value of -4.25. Then from eqn. (3.56) if follows that $\tilde{W}_{22} = 1$. After specifying the remaining two closed-loop values to be the desired values of $\lambda_1 = -4$ and $\lambda_2 = -3$,

$\tilde{W}$ and K can be calculated using the eigenvector/eigenvalue assignment procedure of Section 3.11. This gives

$$K = \begin{bmatrix} -0.75 & -1.75 & 2.50 \\ -1.25 & 2.75 & -4.25 \end{bmatrix}, \qquad W = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 23 & 2.8 & 1 \end{array}\right] \quad (3.71)$$

Figures 3.5 and 3.6 compare the open-loop and closed-loop responses to unit step changes in ξ_1 and ξ_2, respectively, and show that x_1 and x_3 have been made undisturbable.

<u>Example 3.3:</u>

Another 4th order hypothetical state-space system will now be considered to illustrate the design procedure for achieving output undisturbability. The form of the open-loop system is as defined in eqn. (3.1) and the coefficient matrices are:

$$A = \begin{bmatrix} -0.932 & 0.850 & -1.668 & 0.854 \\ -0.060 & -2.018 & 2.060 & -1.030 \\ -1.190 & 0.958 & -1.810 & 0.710 \\ -0.320 & 1.424 & -1.680 & 0.35 \end{bmatrix}$$

$$B = \begin{bmatrix} 5.30 & 1.60 & 1.96 \\ -3.50 & -2.0 & -1.20 \\ -0.50 & -1.50 & 0.60 \\ 1.0 & 0 & 1.80 \end{bmatrix} \qquad D = \begin{bmatrix} 2.4 & -0.04 \\ -3 & 1.3 \\ 0 & 1.3 \\ 1.8 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues of A are -3.0, +0.1, -1.0 and -0.51 and hence the system is open-loop unstable. Also, since A does not contain any zeros the pair (A,D) does not satisfy the conditions for undisturbability defined by expression (3.30) or alternatively by Remark 3.4, and hence all four of its state variables are disturbed by either of the two disturbance variables. Since $y_2=x_3$ the second output variable is disturbable but since y_1 is a linear combination of x_1 and x_2 it is not possible to draw any immediate conclusions as to whether y_1 is disturbable or not.

The control objectives for this example are defined as: (i) to make the closed-loop system stable; (ii) to make both y_1 and y_2 undisturbable by ξ_1 and ξ_2 (this will illustrate the design of a system for *output* undisturbability, cf. Section 3.9); (iii) to specify complex eigenvalues and eigenvectors (just to illustrate that they can be handled directly); and (iv) to use a control law of the form u=Kx, plus feedforward control where necessary.

Because the specification is for output undisturbability and y is not a subset of x, it is necessary to transform the system. The transformation (cf. eqn. (3.50) is

$$z = Q^{-1}x = \begin{bmatrix} C \\ --- \\ E \end{bmatrix} x = \begin{bmatrix} 1 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \qquad (3.72)$$

and the coefficient matrices of the transformed system defined by eqn. (3.51) become

$$Q^{-1}A\,Q \doteq A^{\dagger} = \begin{bmatrix} -0.98 & -0.02 & 0.02 & 0.03 \\ -1.19 & -1.81 & 1.91 & 0.71 \\ -0.06 & 2.06 & -1.97 & -1.03 \\ -0.32 & -1.68 & 1.68 & 0.35 \end{bmatrix} \qquad (3.73)$$

$$Q^{-1}D \doteq D^{\dagger} = \begin{bmatrix} 0 & 1 \\ 0 & 1.3 \\ -3 & 1.3 \\ 1.8 & 0 \end{bmatrix}$$

Comparison of the system characterized by $\{A^{\dagger}, D^{\dagger}\}$ with the conditions in eqn. (3.30), reveals that state variable $z_1=y_1$ is "almost undisturbable" with respect to disturbance ξ_1 since $d^{\dagger}_{11}=0$ and $a^{\dagger}_{12}$, $a^{\dagger}_{13}$ and $a^{\dagger}_{14}$ are very small relative to the other elements of A.

However, feedback control will be required to generate the conditions necessary for complete output undisturbability.

Since $d_{11}^{\dagger}$ and $d_{12}^{\dagger}$ are both zero, the condition implied by eqn. (3.40) is already met by $d_1^{\dagger}$ and hence y_1 and y_2 can both be made undisturbable to ξ_1 without the use of feedforward control. However, feedforward control will be required to give $\ell_2^{\dagger}$ the proper structure to make the system undisturbable to ξ_2.

In the transformed system the number of inputs, r=3, and the specified number of undisturbable states, k=2. Therefore it is possible to *arbitrarily* specify all the eigenvalues and three elements of each eigenvector as follows:

$$\Lambda = \text{diagonal } \{-2+j,\ -2-j,\ -3.0,\ -4.0\} \qquad (3.74)$$

$$W = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1+j & 1-j & 0 & 0 \\ \hline 0.5-j & 0.5+j & 1 & 1 \\ w_{41} & w_{42} & w_{43} & w_{44} \end{array}\right] \qquad (3.75)$$

Note that the unspecified elements w_{ij} are to be calculated as part of the design procedure and that the complex eigenvalues and elements of the eigenvectors must occur as complex conjugate pairs.

It is obvious that W defined by eqn. (3.75) meets the condition for undisturbability specified by eqn. (3.39) and that the system is closed-loop stable. The feedback

controller matrix is then calculated from eqn. (3.53) and is shown in eqn. (3.77) below. The variables $z_1=y_1$ of the closed-loop system are undisturbable by ξ_1 when this feedback controller is used, but a feedforward controller must be designed to make them undisturbable by ξ_2. In order to give $\ell_2^\dagger$ the structure required by eqn. (3.40) it is noted that in this example:

$$\ell_2^\dagger = B^\dagger K^{FF} + d_2^\dagger \tag{3.76}$$

If $\ell_2^\dagger$ is set equal to $\{0,0,1.3,\beta\}$ where the 1.3 is arbitrary then K^{FF} and β can be calculated from eqn. (3.76). Note that the solution for K^{FF} is exact.

The feedback controller $K^\dagger$ is then transformed to correspond to the original system by using the relationship $K=K^\dagger Q^{-1}$ and the final controller matrices are:

$$K = \begin{bmatrix} 0.1342 & 0.4032 & -1.3094 & 0.9886 \\ -0.4469 & 0.5135 & 1.6138 & -0.8569 \\ -2.3555 & -2.6438 & 4.2935 & -2.5016 \end{bmatrix} \tag{3.77}$$

$$K^{FF} = [-0.1231 \quad 0.6308 \quad -0.6923]^T$$

Simulated Results: Figure 3.7 (left) shows the simulated response of y_1 and y_2 to a step change in ξ_1 under both open-loop and closed-loop conditions. As expected the open-loop response of y_1 is essentially zero. The

unstable open-loop response of y_2 is made undisturbable by the feedback controller and undisturbability of y_1 maintained. Note that the closed-loop system has zero offset even though it uses proportional feedback control only; is subjected to a sustained step disturbance; and has no natural integrating modes.

Figure 3.7 (right) shows the simulated system responses to a step change in ξ_2. The open-loop responses of y_1 and y_2 are improved considerably by the feedback controller but as predicted by theory both feedforward plus feedback control is needed to make y_1 and y_2 undisturbable by ξ_2.

3.13 Conclusions

The system property of undisturbability has been formally defined and related to the concepts of uncontrollability, non-accessibility, and structural uncontrollability. Necessary and sufficient conditions for undisturbability have been expressed in terms of both the structure of the coefficients matrices of the state-space model and also in terms of the structure of the system eigenvector matrix. These results provide new insight into the structure of linear multivariable systems and have interesting design implications for the synthesis of regulatory controllers. Specifically the results allow :

(i) determination of the property of undisturbability of specific input-output pairs by a simple inspection of the zero entries in the appropriate matrices, and (ii) they lead to a simple and constructive design procedure for the design of regulatory controllers to achieve undisturbability. An important characteristic of this design procedure is that it makes use of the information contained in the model about the effect of disturbances, whereas other design schemes for feedback controllers do not. In comparison to existing design schemes for disturbance localisation the main advantage of the proposed design procedure is its computational simplicity. The main alternative design scheme besides requiring complex algorithms suffers from the hazards of numerical ill-conditioning.

Undisturbability is closely related to the system property of uncontrollability, and also structural uncontrollability but is not identical. It is also related to observability. The controllers designed to produce output undisturbability can be thought of as modifying the closed-loop system so that disturbance(s) affect only the unobservable modes, i.e. those that do not contribute to the output variables of interest.

Three numerical examples illustrate the applicability of the new theoretical results and the proposed synthesis procedure.

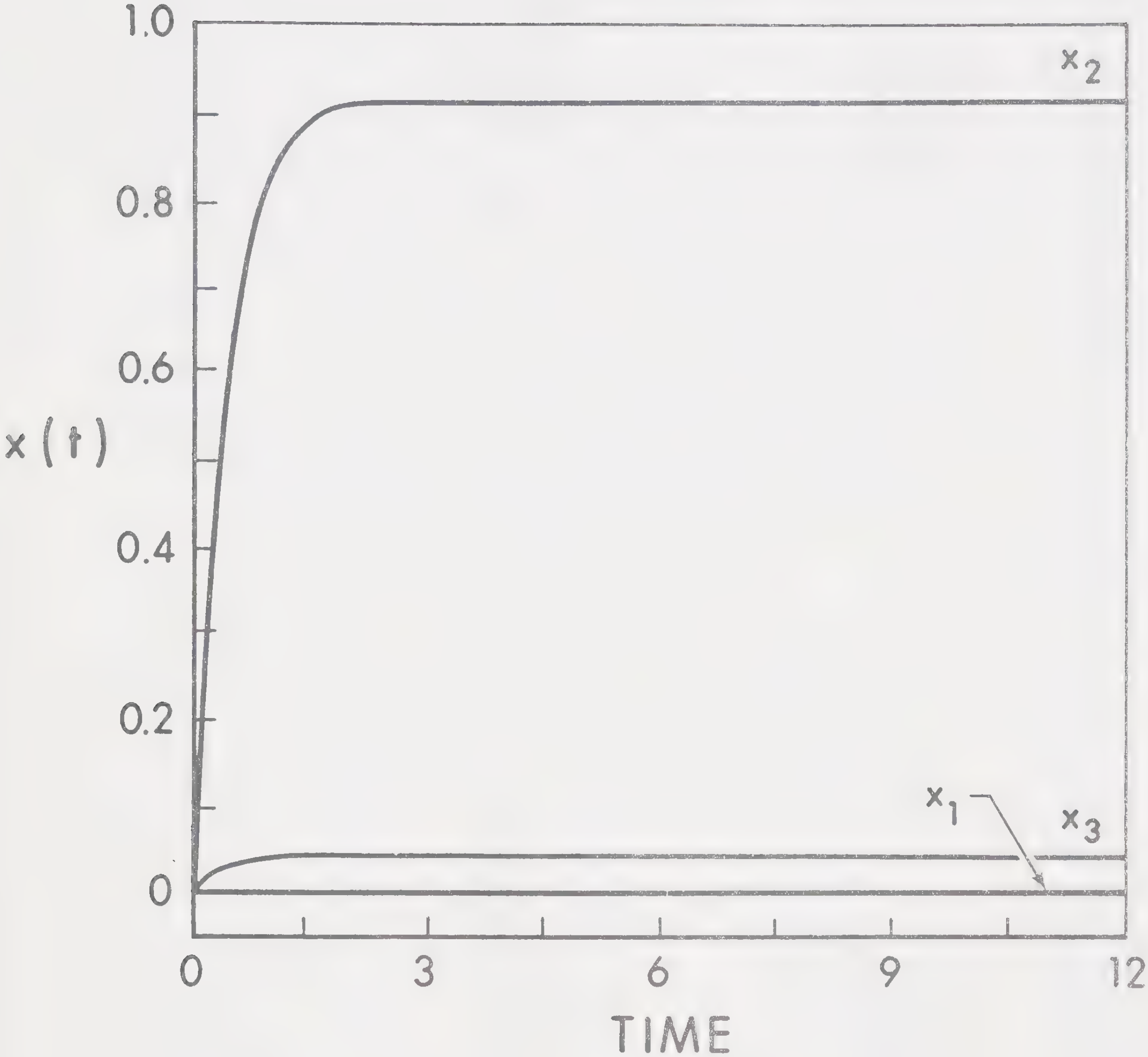

Figure 3.4: Transient responses of the closed-loop system in Example 3.1 to a unit step change in ξ_1.

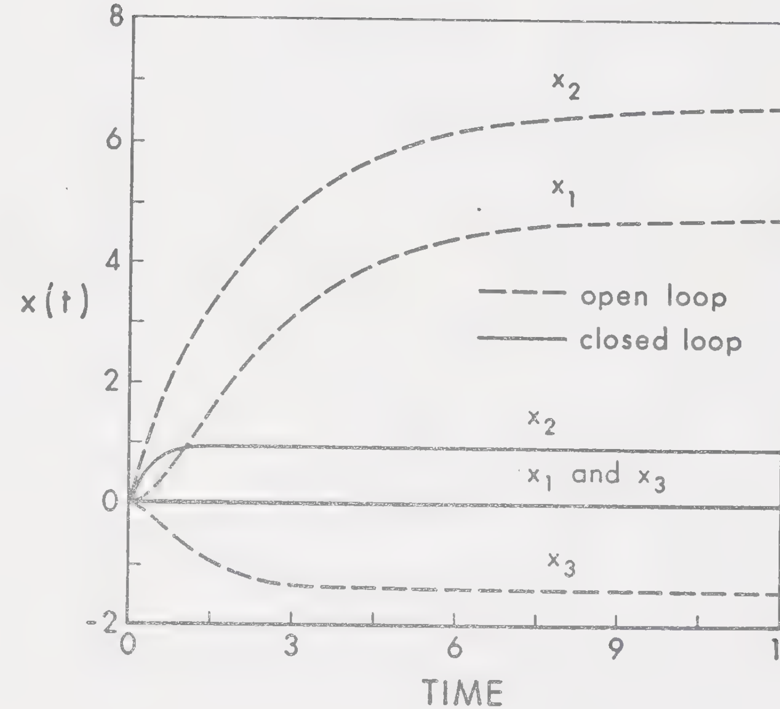

Figure 3.5: Open-loop and closed-loop responses of the system in Example 3.2 for a unit step change in ξ_1.

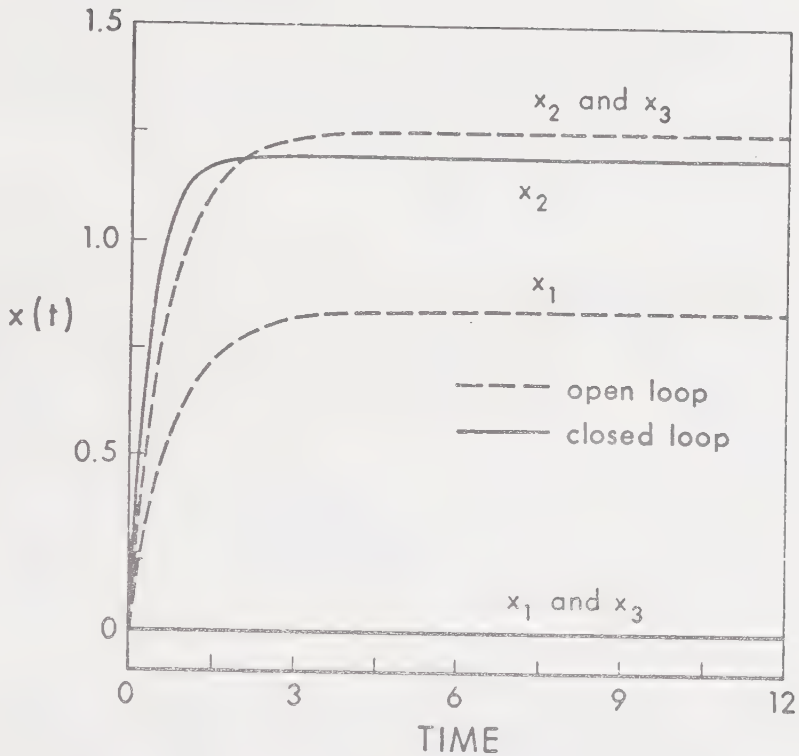

Figure 3.6: Open-loop and closed-loop responses of the system in Example 3.2 for a unit step change in ξ_2.

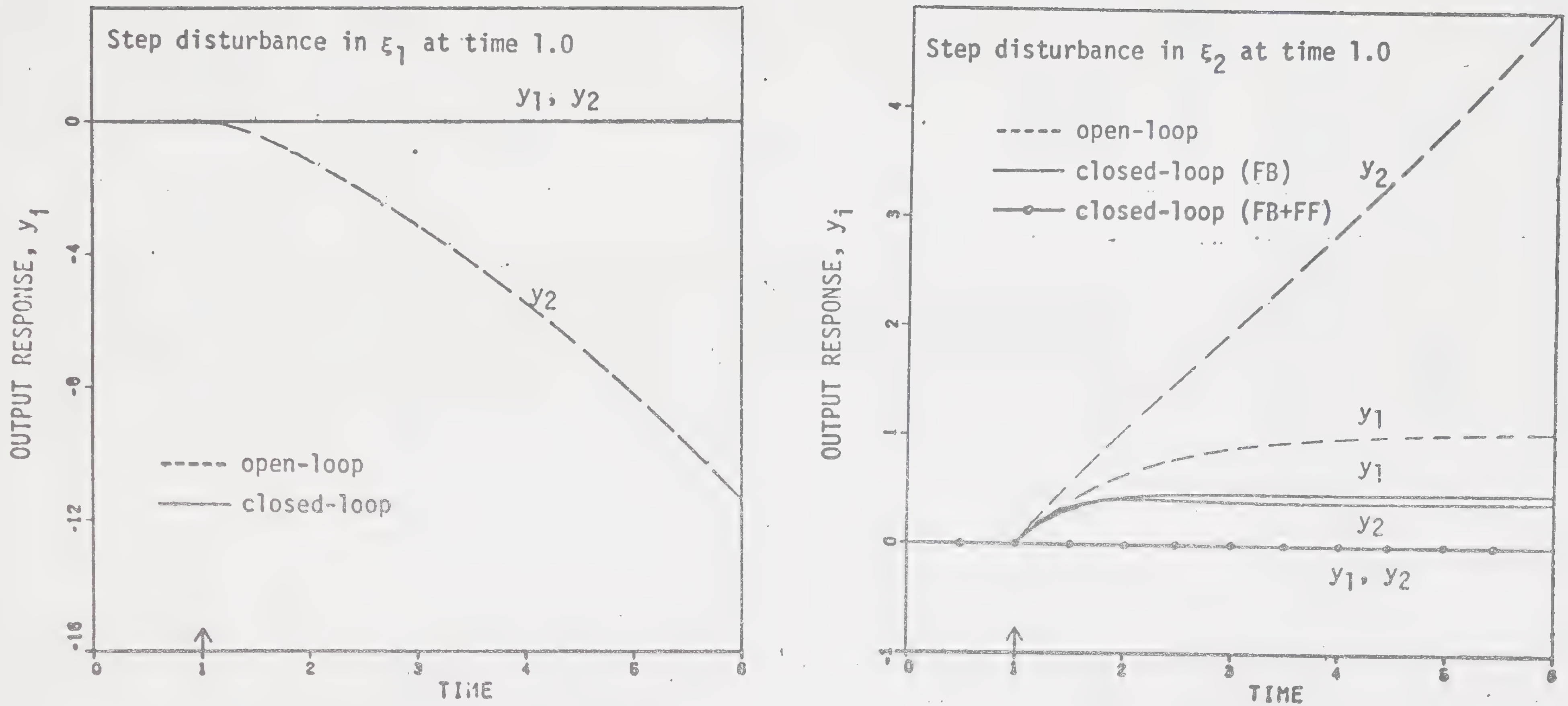

Figure 3.7: Transient responses of the outputs of open-loop and closed-loop hypothetical system of Example 3.3, to unit step disturbances in ξ_1 (left) and ξ_2 (right).

CHAPTER 4

PRACTICAL APPLICATIONS OF MULTIVARIABLE CONTROLLERS DESIGNED TO PRODUCE UNDISTURBABILITY

Abstract

The design procedure for undisturbability developed in Chapter 3 is applied to the design of feedback and feedforward controllers for the 3^{rd} and 5^{th} order models of a double effect evaporator, an 11^{th} order model of an 8 tray binary distillation column, and a 20^{th} order model of another 8 tray, pilot scale, binary distillation column. Experimental application of feedback and feedforward controllers designed to produce undisturbability of the main variables of interest in the computer controlled, pilot-plant evaporator gave results superior to conventional controllers. The problem of simultaneous disturbance localisation and asymptotic set point tracking is also discussed. Some results on disturbance localisation using proportional plus integral feedback are also included.

4.1 Introduction

In this chapter the design procedure for undisturbability, developed in Chapter 3, is applied to the design of multivariable regulators for a double-effect evaporator. Feedback and feedforward multivariable regulators designed to produce undisturbability were evaluated experimentally by application to a computer-controlled, pilot-plant evaporator at the University of Alberta. A detailed description of the evaporator, its models, and a schematic diagram is included in Appendix C. The performance of the evaporator with these multivariable regulators implemented is compared with optimal-quadratic multivariable controllers, and a multiloop scheme that have been applied previously to the same evaporator.

Each control system was evaluated by introducing a step change in one of the disturbance variables of the system. Three different disturbances can be used to upset the pilot-plant evaporator: feed flow, feed concentration and feed temperature. Only the first two have been used in this study. Feed flow has been found to be the most severe disturbance in the system and this was the main reason for using it. The feed concentration disturbance has been used to evaluate the degree of interaction in the control system. The controllers were implemented on

an IBM 1800 digital data acquisition and control computer, which is interfaced to the pilot-plant evaporator, using a computer control package developed by Newell [1].

The simplicity and practicality of the design procedure of Chapter 3 was also demonstrated by its application to the design of multivariable regulators for an 11^{th} and 20^{th} order models of two different distillation columns. The performance of these regulators was evaluated by a digital computer simulation.

Two areas that become important when actual applications were investigated are: (i) provision for making setpoint changes and (ii) possible addition of integral control to the disturbance localisation controllers. While regulatory control in the presence of external disturbances is the prime control objective in most process control systems, asymptotic setpoint tracking or following is also an additional important requirement in an increasing number of supervisory computer control applications. In this chapter the problem of simultaneous disturbance localisation and asymptotic setpoint following is also considered. Because of the quasi-triangular structure of the closed-loop system matrix for undisturbability (cf. Theorem 3.3) and the assumption of nonsingularity of B_1 (cf. Section 3.11), it is shown here that the additional requirement of asymptotic setpoint

following is almost always satisfied. Using these results setpoint controllers were also designed for the double effect evaporator, and the 11^{th} and 20^{th} order models of the two distillation columns. These controllers were evaluated experimentally by application to the evaporator, and by digital simulation of the distillation column models.

Model inaccuracies, parameter fluctuations occur in most practical applications and thus the original design objectives may not be achieved, e.g. in the case of disturbance localisation the state variables of interest may no longer be undisturbable. In such situations the possibility exists that the introduction of integral feedback of the undisturbable state-variables would compensate for the effect of model inaccuracies or unexpected disturbances, especially sustained disturbances. In this chapter such a conjecture is shown to be false.

The present chapter is organised as follows: The results on simultaneous disturbance localisation and asymptotic setpoint tracking are first discussed in Section 4.2. In Section 4.3, the possibility of introducing integral feedback of undisturbable state variables to compensate for the effect of parameter variations, model inaccuracies, and/or sustained unexpected disturbances is shown to be non-existent. The experimental evaluation of multivariable regulators designed to produce undisturbability is discussed in Section 4.4. The application of

the design procedure of Chapter 3, to an 11^{th} and 20^{th} order models of two different distillation columns is considered in Section 4.5 followed by concluding remarks in Section 4.6.

4.2 Simultaneous Disturbance Localisation and Asymptotic Setpoint Tracking

The problem of regulation when a system is subject to external disturbances with simultaneous decoupling was first considered by Mclane and Davison [2], and more recently by Fabian and Wonham [3]. In [3] geometric necessary and sufficient conditions for simultaneous disturbance localisation, decoupling and stabilizability are given. In addition to [3], this problem has also received attention in [4]. The requirement for complete decoupling as well as disturbance localisation results in a set of restrictive necessary and sufficient conditions; steady-state decoupling is a suitably weaker requirement and yields a setpoint controller of constants which is easy to implement. Davison [5] has recently introduced the notion of steady-state invertibility and has derived necessary and sufficient conditions for a system to be steady-state invertible. This section is concerned with the investigation of the problem of simultaneous disturbance

localisation and steady-state decoupling. The main property of interest here is the steady-state decoupling of a quasi-triangular (or undisturbable) closed-loop system characterised by matrices C, B, H and L, where

$$\left.\begin{aligned} C &= \left[\begin{array}{c|c} I_m & 0 \end{array}\right], \qquad B = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] \\ H &= \left[\begin{array}{c|c} H_1 & 0 \\ \hline H_3 & H_4 \end{array}\right] \text{ and } L = \left[\begin{array}{c} 0 \\ \hline L_2 \end{array}\right] \end{aligned}\right\} \tag{4.1}$$

The output vector, y, for the above system is assumed to be of dimension $m \leq r$ (r = number of inputs) and y is also assumed to be a subset of the state vector. Further in eqn. (4.1), $B_1 \in \mathcal{R}^{m\times r}$, Rank $(B_1) = m$, $H_1 \in \mathcal{R}^{m\times m}$, $H_4 \in \mathcal{R}^{(n-m)\times(n-m)}$ and $L_2 \in \mathcal{R}^{(n-m)\times q}$. By application of Theorem 3.3 it is clear that the outputs of the closed-loop system characterised by matrices C, H and L in eqn. (4.1) are undisturbable with respect to the q disturbances $\xi_1, \xi_2, \ldots, \xi_q$. Let the system of eqn. (4.1) be governed by the following control law,

$$u(t) = Kx(t) + K^{SP}y_d(t) \tag{4.2}$$

such that $A + BK \equiv H$. The transfer function between the outputs, $y(s)$, and desired setpoint values, $y_d(s)$, is given by:

$$y(s) = C(sI - H)^{-1}BK^{SP}y_d(s)$$

$$= (sI - H_1)^{-1}B_1K^{SP}y_d(s) \qquad (4.3)$$

From eqn. (4.3) and the final value theorem, it is clear, for the class of inputs where the $\lim_{s \to 0}(sy_d(s))$ is a vector of constant terms, that

$$y(t) = y_d(t) \text{ as } t \to \infty \text{ if:}$$

$$H_1^{-1}B_1\ K^{SP} = - I_m \qquad (4.4)$$

For expression (4.4) to be satisfied with the given assumptions, K^{SP} has to be calculated from the equation:

$$B_1K^{SP} = - H_1 \qquad (4.5)$$

With r = m and B_1 assumed to be nonsingular (cf. Section 3.11), K^{SP} is uniquely defined by

$$K^{SP} = - B_1^{-1}H_1 \qquad (4.6)$$

For r>m, eqn. (4.5) results in a set of (mxm) equations in (rxm) unknowns. Hence, there is no unique solution for K^{SP}. One way to solve for K^{SP} in such a case is to assign arbitrary values to (rxm - mxm) parameters in K^{SP} and then solve for the remaining (mxm) unknowns.

Alternatively out of r available control inputs one could choose the most suitable m inputs and thus solve for a unique K^{SP}. For the case where $r<m$, i.e. the number of outputs is greater than the inputs, eqn. (4.5) cannot be satisfied exactly. However, such a case is not very common in control applications.

It is clear from eqn. (4.5) that for the more general case where $r \geq m$, and where undisturbability of the m output variables has been achieved, i.e. H_1^{-1} exists by design, and Rank $(B_1) = m$, then simultaneous disturbance localisation and asymptotic tracking of a certain class of inputs is always possible. Design examples to demonstrate degree of decoupling achieved using this procedure will be considered in Sections 4.4 and 4.5.

4.3 Disturbance Localisation Using Proportional Plus Integral Feedback

Model inaccuracies and parameter variations occur in most practical applications and thus the original design objectives for which a state feedback controller may have been designed e.g. for undisturbability, may not be realised. In such situations the possibility exists that the introduction of integral feedback of the undisturbable state variables would compensate for the

effect of model inaccuracies, and/or parameter variations, and/or sustained unexpected disturbances. In the following analysis it is shown that theoretical difficulties arise.

The design procedure for undisturbability as developed in Chapter 3 is based on eigenvector/eigenvalue assignment. Moore [6] has recently characterised the class of all closed-loop eigenvector sets which can be obtained with a given set of distinct closed-loop eigenvalues, by providing conditions which the closed-loop system eigenvectors must satisfy (cf. Section 2.2). In the ensuing analysis use is made of these conditions to show why integral feedback of the undisturbable state variables is not possible when $H_2 = 0$.

Integral feedback compensation will be considered by augmenting the original state vector x(t) with a ℓx1 vector z(t) where z(t) is defined by:

$$\dot{z}(t) = T\, x(t) \tag{4.7}$$

In eqn. (4.7) $T \in R^{\ell x n}$, and it consists of ℓ appropriate rows to include a subset of state variables or a linear combination of state variables requiring integral feedback. Note that because of controllability conditions on the augmented system, $\ell \leq r$. Let $\hat{A}$ and $\hat{B}$ now denote the augmented system state matrix and input matrix, respectively. Let the pair $(\hat{A},\hat{B})$ be first partitioned in the form:

$$\left[\begin{array}{c|cc} A_{11} & A_{12} & 0 \\ \hline A_{21} & A_{22} & 0 \\ T_1 & T_2 & 0 \end{array}\right], \quad \left[\begin{array}{c} B_1 \\ \hline B_2 \\ 0 \end{array}\right] \tag{4.8}$$

Where $A_{11} \in R^{kxk}$, $A_{22} \in R^{(n-k)x(n-k)}$, $T_1 \in R^{\ell xk}$ and $B_1 \in R^{kxr}$. If the state variables $x_1, x_2, \ldots, x_k$ are to be made undisturbable then the structure of the closed-loop system matrix H and the corresponding eigenvector matrix W must be of the form (cf. Theorem 3.3 and 3.5):

$$H = \left[\begin{array}{c|cc} H_{11} & 0 & 0 \\ \hline H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{array}\right] \leftrightarrow W = \left[\begin{array}{c|cc} W_{11} & 0 & 0 \\ \hline W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{array}\right] \tag{4.9}$$

In (4.9) matrices H and W have been partitioned so as to be compatible with the partitions of A in (4.8). The design procedure for eigenvalue/eigenvector assignment requires that the closed-loop system eigenvalues be distinct which in turn implies that the closed-loop system eigenvectors must be linearly independent. This requirement will play a key role in the following

investigation. The main result of this study with respect to integral control is this: *The closed-loop eigenvectors of an undisturbable system with state plus integral feedback will not be linearly independent* <u>*if*</u> *Rank* $(T_2) \neq \ell$, *where* T_2 *has been defined in (4.8).*

<u>Remark 4.1</u>: From an inspection of expression (4.8) the main consequence of the above result is this: If integral feedback of an undisturbable state variable, x_i $(i \leq k)$, is required then the $(i,i)^{th}$ $(i \leq \ell)$ element of T_1 should be equal to unity; however, for the design procedure to be possible the closed-loop eigenvectors must be linearly independent, i.e. Rank $(T_2) = \ell$, which implies that the i^{th} row of T_2 should at least have one nonzero term in it. This means that integral feedback of x_i *with* a linear combination of other state variables $x_j (k<j \leq n)$ is possible, and integral feedback of x_i *alone* is not possible.

The proof for the above result proceeds in the following way. For the eigenvectors in (4.9) to be linearly independent, W^{-1} must exist. Since W is quasi-triangular this implies that W_1^{-1} and W_4^{-1} must exist, where using the notation of Theorem 3.5, W_1 and W_4 are defined by:

$$W_1 = W_{11}\ , \qquad W_4 = \begin{bmatrix} W_{22} & W_{23} \\ W_{32} & W_{33} \end{bmatrix} \tag{4.10}$$

Now for $k \leq r$, W_1 can be chosen arbitrarily and thus existence of W_1^{-1} can be assured. For W_4^{-1} to exist it is necessary that the ℓ rows formed by $[W_{32} \;\; W_{33}]$ be linearly independent. To find conditions under which Rank $[W_{32}\ W_{33}] = \ell$, consider the class of eigenvectors that characterise W_4 by using the result of Moore [6], i.e. the i^{th} eigenvector, w_i, must satisfy the equation:

$$\left[\begin{array}{ccc:c} \lambda_i I - A_{11} & -A_{12} & 0 & B_1 \\ -A_{21} & \lambda_i I - A_{22} & 0 & B_2 \\ -T_1 & -T_2 & -\lambda_i I & 0 \end{array}\right] \begin{bmatrix} w_i^1 \\ w_i^2 \\ w_i^3 \\ M_{\lambda_i} \end{bmatrix} = 0 \tag{4.11}$$

In eqns. (4.11) w_i corresponds to λ_i and M_{λ_i} is defined by eqn. (2.9). For $k < i \leq n + \ell$, $w_i^1 = 0$ (cf. eqn. (4.9)). This means that w_i^2 and w_i^3 for all $k < i \leq n + \ell$ must satisfy the equation:

$$-\lambda_i I w_i^3 = T_2 w_i^2 \tag{4.12}$$

Equation (4.12) clearly implies that Range $[W_{32} \;\; W_{33}] \subseteq$ Range T_2. Then Rank $[W_{32} \;\; W_{33}] = \ell$ only if Rank $T_2 = \ell$.

Thus if Rank $T_2 \neq \ell$, then Rank $[W_{32} \;\; W_{33}] < \ell$ which implies that W_4^{-1} does not exist. This completes the proof.

The important point to note is that T_2 can have no zero rows. This means that it is not possible to have independent integral feedback control of an undisturbable variable by using eigenvector assignment techniques. For example, if x_i $(i \leq k)$ is to be made undisturbable and it is also desired to have an integral feedback of x_i alone, then this is not possible. However, with x_i undisturbable integral feedback of $(x_i + \sum_{j=k+1}^{n} x_j \alpha_j)$, with at least one $\alpha_j \neq 0$ is possible. If H_2 is made non-zero (eg. to indicate parameter variations in A_2, B_1 or K_2) then it is possible to design an integral controller for x_i alone.

4.4 Evaporator Application

The design procedure for undisturbability developed in Chapter 3 was evaluated by application to the computer-controlled, pilot-plant evaporator at the University of Alberta. This evaporator has been used in a number of other studies to evaluate different modern multivariable control techniques (Fisher and Seborg, [7]) and hence it is possible to make direct comparisons between alternative control techniques.

A description of the pilot-plant evaporator and its models is given in Appendix C. A schematic diagram of the evaporator and the conventional multiloop control scheme used as the 'base case' for comparison is also shown in Figure C.1, Appendix C. The primary control objective is to keep the product concentration, C2, constant despite disturbances in the feed flowrate, F, the feed concentration, CF, and/or the feed enthalpy, HF. It is also necessary to keep the two liquid holdups W1, and W2, within operating limits, but small variations in these variables are acceptable. The control (manipulated) variables are the steam flowrate, S, and the bottom product flowrate from each of the two effects, B1 and B2. In summary, the evaporator has 3 output variables, 3 control variables and 3 disturbance variables. These variables and their normal operating steady state values are defined in Appendix C.

A number of different models of the evaporator have been derived in previous studies (e.g. Newell and Fisher, [8]). The models used in this study are the 3rd and 5th order, discrete, state-space models defined by the coefficient matrices listed in Tables C.2 and C.3. The 3rd order model is not as accurate as the 5th and 10th order models used in other studies, but is in reasonable agreement with experimental data and makes

it much easier to follow the design procedure. Hence, the design of a controller to produce undisturbability will be first considered for this 3rd order model.

4.4.1 Controller design for the 3rd order model

The 3rd order model has as its state variables, W1, W2 and C2; as its input variables S, B1 and B2; and as its disturbance variables, F, CF, and HF. Previous operating experience has shown that the most frequent and severe disturbances in product concentration, C2, are produced by variations in feed flowrate, F. Therefore for purposes of this application the design objectives for the controller are:

i) to make C2 undisturbable by F.

ii) to assign all closed-loop eigenvalues closer to the origin.

iii) to preserve the open-loop undisturbability of W1 and W2 by CF.

The linearised model in the discrete form as used in this study can be represented by:

$$x(n+1) = \Phi x(n) + \Delta u(n) + \theta d(n)$$

with the coefficient matrices, Φ, Δ and θ and elements of x, u and d given in Appendix C.

According to the 3^{rd} order model, the evaporator, under open-loop conditions, is only conditionally stable since the eigenvalues of Φ are 0.96, 1 and 1. The two unit eigenvalues in Φ follow directly from the "integrating nature" of the two liquid holdups. Inspection of the coefficient matrices for the 3^{rd} order model indicates that under open-loop conditions, all three state variables are disturbable with respect to F and HF, while W1 and W2 are undisturbable with respect to CF. (As noted in Remark 3.7 of Chapter 3, the results in Section 3.7 can easily be extended to discrete-time systems.)

Since $\theta_{31} \neq 0$ it is not possible to make C2 undisturbable with respect to F by feedback control alone. However, a feedback feedforward control law of the form:

$$u(n) = K\, x(n) + K^{FF} F(n)$$

is required. In the discrete form of the closed-loop system equation, $L = \theta + \Delta K^{FF}$, and hence it is possible by feedforward control to produce the required structure in L. Setting $\ell_1^T = [\alpha, \beta, 0]$ with α and β arbitrarily chosen as 0.12 and 0.0032 gives, K^{FF1} as:

$$K^{FF1} = \begin{bmatrix} 0.32 \\ -0.12 \\ -0.55 \end{bmatrix}$$

Note that if α and β were chosen as zero, than all three state variables would be undisturbable with respect to F. This is possible because in this particular example Δ is non-singular and consequently the equation, $\ell_1 = \theta_1 + \Delta K^{FF}$, has an exact solution for K^{FF}. To complete the controller design, the feedback matrix K is calculated using the eigenvector/eigenvalue assignment procedure described in Chapters 2 and 3. Since in this example, r=n, it is possible to assign all elements of the eigenvector matrix W. If W is chosen to be the same as the open-loop eigenvector matrix (a diagonal matrix since Φ is diagonal), then the following design objectives are realized:

i) C2 is undisturbable with respect to F since W and ℓ_1 are in the form of eqns. (3.39) and (3.40).

ii) W1 and W2 are undisturbable with respect to CF (as was the case for the open-loop system).

Thus choosing W to be the open-loop eigenvector matrix, i.e. W = I, and assigning the closed-loop eigenvalues to be 0.28, 0.47 and 0.65 leads to the following state feedback control matrix; $K_{d\ell 1}$:

$$K_{d\ell 1} = \begin{bmatrix} 2.70 & 0 & -9.69 \\ 3.23 & 0 & 3.88 \\ 4.29 & 13.05 & 17.17 \end{bmatrix} \qquad (4.13)$$

Table 4.1 contains a summary of this and other controllers used in the experimental evaluation.

4.4.2 Signal flow graph analyses

The signal flow graph in Figure 4.1 (left) shows that in the simplified, 3^{rd} order evaporator model, all three states are accessible to a disturbance in feedflow, F, but only the product concentration is accessible by a disturbance in feed concentration, CF. In other words W1, W2 and C2 are disturbable by F and W1 and W2 are undisturbable by CF.

Figure 4.1 (centre) is the signal flow-graph of the closed-loop evaporator system using the disturbance localisation controller discussed above. Comparison with the open-loop system flow graph shows that the feedforward controller eliminates the path from F to C2 and hence makes C2 inaccessible and undisturbable by F.

Figure 4.1 (right) is the signal flow graph of the evaporator system using the basic multiloop control scheme shown in Fig. C.1. The multiloop controller,

$K_{m\ell}$, used in this comparison is shown in Table 4.1. Note that in using this controller all states become accessible to both disturbances, and that a number of interactions are introduced when previously there were none. This emphasizes the importance of examining the structure of open and closed-loop system matrices.

4.4.3 Experimental results

Figure 4.2 shows the response of the evaporator to $\mp$ 30% step changes in feed concentration, introduced at the times indicated by the arrows on the time axis. Controller, $K_{d\ell 1}$, was designed to retain the open-loop undisturbability of the liquid levels to concentration changes and the results in Figure 4.2 confirm that there is no significant effect of CF on W1 or W2. The controller was not designed to make C2 undisturbable by CF and a small offset in C2 is obvious in Figure 4.2. Note that this controller is an ideal modal controller since the open-loop and closed-loop eigenvectors are identical.

The evaporator response to $\mp$ 30% changes in feed concentration using the multiloop controller, $K_{m\ell}$, is shown in Figure 4.3. This experimental data confirms the conclusion derived from the signal flow graph analyses,

that the multiloop control scheme introduces significant interactions.

Figure 4.4 shows the evaporator response to ±20% step disturbances in feed flowrate using the feedback plus feedforward controller, $K_{d\ell 1}$ plus K^{FF1}. Figure 4.5 shows the evaporator response to ±20% step disturbance in F using feedback control, $K_{d\ell 1}$, alone (equivalent to the case where disturbance measurements are unavailable). Another controller, $K_{d\ell 2}$, which retains the same open-loop eigenvector structure but assigns eigenvalues at 0.59, 0.43 and 0.34 was also designed. The response of the evaporator with a ±20% disturbance in feed flowrate using only the feedback controller $K_{d\ell 2}$ is shown in Figure 4.6. Comparison of Figures 4.5 and 4.6, and controllers $K_{d\ell 1}$ and $K_{d\ell 2}$ serves to illustrate the design freedom available, i.e. arbitrary eigenvalue/eigenvector assignment.

Figures 4.7 and 4.8 show the evaporator response to the same disturbance using (i) the multiloop controller $K_{m\ell}$ and (ii) the feedback plus feedforward controller, $K_{m\ell}$ and K^{FF2} as shown in Table 4.1. Also listed in Table 4.1 is a third order optimal proportional controller, K_{opt1}. For the purposes of comparison with Figures 4.3 to 4.8, two experimental runs were performed, using this optimal controller. The results for a ±20%

disturbance in feed flowrate are plotted in Figure 4.9 and Figure 4.10 shows the evaporator response to a $\mp 30\%$ disturbance in feed composition.

The response of the evaporator to a $\pm 20\%$ disturbance in feed flowrate or a $\mp 30\%$ disturbance in feed composition using controller, $K_{d\ell 1}$ or $K_{d\ell 2}$ was superior than the performance of the multiloop controller, $K_{m\ell}$, and better than, or at least comparable to, the performance of optimal controller, K_{opt1}.

With the feed forward control mode implemented the overall performance of the controller $K_{d\ell 1}$ plus K^{FF1} is better than the performance of $K_{m\ell}$ and K^{FF2}. It should be noted that when implementing feedforward control with the multiloop control configuration, $\ell_1 \equiv \theta_1 + \Delta K^{FF2} = 0$, that is the signal flow paths from F to W1, W2 and C2 have all been 'cancelled out' by the effect of control. In contrast to this the feedforward control action in conjunction with $K_{d\ell 1}$ or $K_{d\ell 2}$ requires that only the signal flow path from F to C2 be 'cancelled out'. The net result of this is that feedforward controller, K^{FF2}, with the multiloop control configuration must generate more control action, i.e. use higher gains than the feedforward controller, K^{FF1}. This is also evident from a comparison of the steam, S, to feed flowrate, F, gains in K^{FF1} and K^{FF2}.

In general, the design procedure for 'undisturbability' was found to be practical, was easy to use, and performed well experimentally.

4.4.4 Controller design for the 5^{th} order model

The main reason for using the 5^{th} order model of the evaporator was to demonstrate the use of the design procedure of Chapter 3, for the general case where the number of inputs, r, is less than the number of state variables, n. Application of the design procedure to the 5^{th} order model is also useful in demonstrating the use of the stability criterion for disturbance localisation, i.e. to check if the closed-loop system is stable (cf. Section 3.11.2). A discrete 5^{th} order model of the evaporator is shown in Table C.2. The state variables for the 5^{th} order model are: W1, C1 (first effect composition), H1 (first effect enthalpy), W2 and C2. The control inputs, disturbances and the outputs are identical to those for the 3^{rd} order model. The main control objective as before is to make C2 and W2 undisturbable with respect ot disturbance F and also to retain the open-loop undisturbability of W1 and W2 to CF.

As a preliminary first step the state vector x

was reordered as $\tilde{x}^T = [C2, W2, W1, C1, H1]^T$ and the state, control and disturbance matrices were permuted accordingly and designated by $\tilde{\phi}$, $\tilde{\Delta}$ and $\tilde{\theta}$. Since the first three rows of $\tilde{\Delta}$ are linearly independent it is possible to assign arbitrary entries in the first three rows of the closed-loop system eigenvector matrix using the eigenvalue/eigenvector assignment technique. In order to realise the above mentioned control objectives the following values of $\tilde{W}_1$ and $\tilde{W}_2$ were selected:

$$\tilde{W}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \tilde{W}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.14}$$

Observe that since $\tilde{W}_2 = 0$ it is not possible to assign eigenvalues corresponding to these two eigenvectors (cf. Section 3.11.2). This necessitates use of the stability criterion to check if the resulting closed-loop system is stable. Equation (3.57) shows that for this case the matrix P is:

$$P = \begin{bmatrix} 0.90 & 0 \\ -0.31 & -0.76 \end{bmatrix} \tag{4.15}$$

and thus has eigenvalues: 0.9 and -0.76 which are inside the unit circle. This means that two of the closed-loop

system eigenvalues are -0.76 and 0.9 while the remaining three can be chosen arbitrarily as 0.28, 0.47 and 0.65. The final closed-loop eigenvector matrix $\tilde{W}$ was evaluated from eqns. (3.55) and (3.56) and is

$$\tilde{W} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0.62 & 0 & -0.43 & 0 & 0.98 \\ -5.33 & 0 & 1.12 & 1 & -.19 \end{array}\right] \tag{4.16}$$

$\tilde{W}$ as shown above is non-singular and in the required form for undisturbability (cf. Theorem 3.5). The feedback controller $K_{d\ell 5}$, was then calculated using the pseudoinverse of $\tilde{\Delta}$ and is shown in Table 4.1. The resulting closed-loop system matrix is:

$$\tilde{\phi} + \tilde{\Delta}\,\tilde{K}_{d\ell 5} = \left[\begin{array}{ccc|cc} .28 & 0 & 0 & 0 & 0 \\ 0 & .47 & 0 & 0 & 0 \\ 0 & 0 & .65 & 0 & 0 \\ \hline -.40 & 0 & .11 & .90 & 0 \\ -5.35 & 0 & 1.45 & -.31 & -.76 \end{array}\right] \tag{4.17}$$

Since $\tilde{\theta}_1$ does not have zero entries in appropriate places a feedforward controller is also required to complete the design requirements for undisturbability.

For the purposes of illustration, the design of two different feedforward controllers K^{FF3} and K^{FF4} was considered in this application. K^{FF3} was designed to make C2 and W2 undisturbable with respect to F (i.e. $\tilde{\ell}_1^T =$ $[0 \quad 0 \quad \alpha \quad \beta \quad \gamma]$), and K^{FF4} was designed to make all three output variables C2, W2 and W1 undisturbable with respect to F (i.e. $\tilde{\ell}_1^T = [0 \quad 0 \quad 0 \quad \nu \quad \mu]$). These two feedforward controllers are defined in Table 4.1.

4.4.5 Experimental results

Figures 4.11, 4.12 and 4.13 show the evaporator responses to a ±20% step disturbance in feed flowrate with feedback controller, $K_{d\ell 5}$, implemented, with $K_{d\ell 5}$ plus K^{FF3} implemented, and with $K_{d\ell 5}$ plus K^{FF4} implemented. For comparison purposes the evaporator response to a ±20% disturbance in feed flowrate using a 5th order proportional optimal controller is also shown in Figure 4.14. The evaporator performance using any one of these 5th order state feedback controllers implemented in each of the four experimental runs is better than that obtained using a 3rd order controller. This would be expected because the 3rd order model is not as accurate as the 5th order model. From an overall performance point of view the evaporator response to feed disturbances using

feedback controller, $K_{d\ell 5}$, plus feedforward controller, K^{FF3}, or that obtained using only feedback controller, $K_{d\ell 5}$, is at least comparable to or even better than the proportional optimal control. Another 5^{th} order controller, K_{16}, as designed by using a simultaneous eigenvalue assignment procedure (Park [9]) was also implemented on the evaporator. Some difficulty was experienced in implementing this controller. Its performance as shown in Figure 4.15, was found to be oscillatory. Difficulty was experienced in implementing other controllers designed by the same method. A 3^{rd} order modal controller, G_{19}, based on Rosenbrock's approximate modal control method [10] was also implemented on the evaporator. The performance of this controller to step disturbances in the feed flowrate, as shown in Figure 4.16, was satisfactory but not as good as the response of controller, $K_{d\ell 1}$, as shown in Figure 4.5.

4.4.6 Design of setpoint controllers for the double-effect evaporator

The 3^{rd} order evaporator model has 3 state-variables, 3 inputs and 3 outputs. Because of the discrete nature of the evaporator models, with only minor modifications, the results of Section 4.2 can be applied

directly to calculate the setpoint controller, K^{SP1}, for the discrete 3^{rd} order evaporator model. The condition for asymptotic setpoint tracking or following in discrete systems for the 3^{rd} order evaporator model requires that:

$$(I - (\Phi + \Delta K_{d\ell 1}))^{-1} \Delta K^{SP1} = I \qquad (4.18)$$

The resulting, K^{SP1}, for the 3^{rd} order model is listed in Table 4.1. For the discrete 5^{th} order evaporator model, K^{SP3}, is given by:

$$K^{SP3} = \tilde{\Delta}_1^{-1} (I - \tilde{H}_1) \qquad (4.19)$$

where $\tilde{H}_1$ is the upper left 3 x 3 matrix partition of $\tilde{H}(= \tilde{\Phi} \quad \tilde{\Delta} \tilde{K}_{d\ell 5})$ as shown in eqn. (4.17); $\tilde{\Delta}_1$ is the upper 3 x 3 matrix partition of $\tilde{\Delta}$ with its first three rows corresponding to state variables C2, W2 and W1. The resulting setpoint controller, K^{SP3}, for the 5^{th} order model is listed in Table 4.1. The evaporator response to a ±10% step change in the setpoint of C2 with controllers $K_{d\ell 1}$ plus K^{SP1}, and $K_{d\ell 5}$ plus K^{SP3} implemented is shown in Figures 4.17 and 4.18. As expected the evaporator response to ±10% step change in the setpoint of C2 with controller $K_{d\ell 5}$ plus K^{SP3} was better than that of the 3^{rd} order controller, $K_{d\ell 1}$ plus K^{SP1}. It should be noted from Figures 4.17 and 4.18,

that in each case the asymptotic setpoint tracking property of controllers K^{SP1} and K^{SP3} is approximately satisfied. For the purpose of comparison, the evaporator response to a ±10% step change in the setpoint of C2 using an optimal controller, K_{opt1} plus K^{SP2}, was also evaluated experimentally and is shown in Figure 4.19. The evaporator response in Figures 4.17 and 4.18 is better than or at least as good as that in Figure 4.19.

In this section results on experimental evaluation of the design procedure for undisturbability have been presented. In general the design procedure for undisturbability was found to be practical and convenient. It was easy to use, gave considerable insight into system performance, placed the closed-loop eigenvalues and eigenvectors in the desired locations, produced practical controllers with reasonable gains, and performed very well experimentally. To the best of the author's knowledge, this is the first reported application of this design approach.

Experimental evaluation of controllers, $K_{d\ell 1}$ plus K^{SP1}, and $K_{d\ell 5}$ plus K^{SP3} also confirmed the results of Section 4.2 that simultaneous disturbance localisation and asymptotic setpoint following is possible. The design procedure for calculating such setpoint controllers

is easy to use and gives easily implementable constant setpoint controllers.

4.5 Distillation Column Application

4.5.1 11^{th} order distillation column model of Davison [11]

A state space model of an 8 tray binary distillation column with variable column pressure has been derived by Davison [11]. The linearized equations of the 11^{th} order system with 3 inputs, 3 outputs and 1 main disturbance are as follows:

$$\dot{x}(t) = A\,x(t) + B\,u(t) + Dw_m(t), \quad y(t) = C\,x(t) \tag{4.20}$$

The state vector, x(t), and the output vector, y(t), are:

$$x^T = (x_c, x_1, x_2, \ldots, x_8, x_b, p) \quad \text{and}$$

$$y^T = (x_c, x_b, p)$$

with x_c = composition of more volatile component in condenser (mole fraction); x_i = composition of more volatile component on plate i, i = 1,2, ..., 8 (mole fraction); x_b = composition of more volatile component in reboiler (mole fraction); u_1 = reboiler temperature; u_2 = condenser temperature; u_3 = reflux flow (lb-mole/sec); p =

pressure in the system (atm) and w_m = feed composition. Numerical values of matrices A, B and D are given in [12].

According to Davison [12], the main control objective in the design of a controller for the column is to regulate the top (y_1), and bottom product (y_2) compositions, and the pressure (y_3) in the presence of composition fluctuations in the input feed stream. The open-loop distillation column is stable. Inspection of matrices A, B and D reveals the following:

i) Elements of D corresponding to state variables x_c, x_b and p are zero, hence no feedforward control will be required in the design for undisturbability.

ii) The row of B corresponding to state variable x_c has all zero elements. Consequently the corresponding partition B_1 is singular. This means that the design technique for undisturbability cannot be applied directly, and hence the control objectives have to be modified.

iii) In modifying the control objectives use is made of the following important characteristic of the system: that is since x_c cannot be influenced *directly* by any one of the three inputs,

the corresponding row of A will also be the row of the closed-loop system matrix, H. Inspection of the first row of A corresponding to x_c also reveals that, x_c is accessible to the control inputs via a coupling between x_c and x_1. This is important because it means that if x_1 is made undisturbable then x_c will also become undisturbable, especially since the first row of A is invariant under any feedback. Thus the modified control objective will be to make x_1, x_b and p undisturbable with respect to w_m. Then once x_1 becomes non-accessible to w_m, undisturbability or nonaccessibility of x_c to w_m can also be assured. Observe that the first three rows of $\tilde{B}$, as denoted by $\tilde{B}_1$ corresponding to the modified system with $x = [\tilde{x}_1, x_b, p, x_c, x_2, x_3, \ldots, x_8]$ are now linearly independent. The rest of the design procedure is now simple. With a choice $\tilde{W}_1 = I_3$, and $\tilde{W}_2 = 0$ (cf. Theorem 3.5), and Λ_1 = diag. (-.3,-.25,-.2), a check is made to see if the eigenvalues of matrix P (cf. eqn. (3.57)) are in the left half of the complex plane (Section 3.11.2). The eigenvalues of P do indeed turn out to be in the left half of the complex plane. With this check complete, $\tilde{K}$ is very simply evaluated from the equation:

$$\tilde{K} = \tilde{B}_1^{-1}([\tilde{A}_1 \mid \tilde{A}_2] - [\tilde{H}_1 \mid 0]), \text{ with } H_1 = \Lambda_1 \tag{4.21}$$

Notice that computation of $\tilde{K}$ only requires inversion of a 3x3 matrix, and matrix multiplication. The computed $\tilde{K}$ was found to be:

$$\tilde{K} = \begin{bmatrix} -4271.6 & 2975.7 & -247.3 & -176.5 & -68.7 & 0 & 0 & 0 & 0 & 0 & 301.5 \\ 4271.6 & -3031.1 & -147.2 & 121.1 & 68.7 & 0 & 0 & 0 & 0 & 0 & -301.5 \\ -37.6 & -54.5 & -2.1 & -1.5 & -0.6 & 0 & 0 & 0 & 0 & 0 & -5.4 \end{bmatrix} \tag{4.22}$$

The resulting closed-loop system matrix, $\tilde{H} = \tilde{A} + \tilde{B}\tilde{K}$ and the disturbance matrix $\tilde{D}$ are shown in Table 4.2. Both $\tilde{H}$ and $\tilde{D}$ are in the required form for x_1, x_b, p and x_c to be undisturbable. Figure 4.20, shows the response of x_c, x_b and p to a unit step disturbance in w_m. Also superimposed on each of plots in Figure 4.20, is the corresponding open-loop system response to a step disturbance in w_m. For the purpose of comparison an attempt was also made to reproduce the response of the distillation column with a conventional controller and another controller designed by Davison [12]. However, difficulties* were encountered in reproducing these two results, and hence each plot in Figure 4.20 has an insert from [12] showing the nature of these responses. For convenience the controllers depicted as 'conventional control' and 'new control system' in the figure caption in the inserts are also reproduced here:

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* Use of the numerical values in [12] led to unstable responses.

New control system [12]:

$$u_1 = -1000\,x_2 + 600\,x_7 - 1000\,P - 45.2\,w_m + 9010\,\dot{w}_m$$

$$u_2 = 1000\,x_2 - 600\,x_7 - 1000\,P - 223\,w_m + 49900\,\dot{w}_m$$

$$u_3 = -1000\,x_2 - 1000\,x_7 - 286\,w_m + 64900\,\dot{w}_m$$

Conventional control system [12]:

$$u_1 = 100{,}000\,x_7$$

$$u_2 = -100{,}000\,p$$

Notice that Davison's controller requires feedforward control of w_m and also computation of $\dot{w}_m$

Using the results of Section 4.2, a setpoint controller, $\tilde{K}^{SP}$, was also designed for the above distillation column. This was calculated to be:

$$\tilde{K}^{SP} = \begin{bmatrix} 14578.2 & -3731.3 & 272.6 \\ -14578.2 & 3731.3 & 162.2 \\ 128.3 & 67.2 & 2.1 \end{bmatrix} \tag{4.23}$$

Implementation of $\tilde{K}^{SP}$ as shown above results in an almost completely non-interacting system as evident by the response of the distillation column to unit step setpoint changes in each of the variables x_c, x_b and p (Figure 4.21).

4.5.2 20^{th} order distillation column model of McGinnis [13]

McGinnis [13] has derived a 20^{th} order linear state space model of a pilot scale distillation column at the University of Alberta. This model has been derived by considering stage enthalpies and liquid flows at each of the ten stages. A reduced 2^{nd} order model has also been obtained from this higher order model, and has been used in previous control applications, Wood and McGinnis [14]. The 20^{th} order model has 2 inputs, 2 outputs and 3 disturbances. The outputs of interest are the liquid phase enthalpy in the reboiler, $h_1 (=y_1)$, and the liquid phase enthalpy in the condenser, h_2 $(=y_{10})$. The 3 disturbances of interest are feed flowrate, feed enthalpy and condenser heat duty. Of these, feed flowrate has been found to be the most severe disturbance and hence the main control objective in this study is to make the two state variables or the outputs, h_1 and h_{10}, undisturbable with respect to this disturbance. The 2 control variables are reflux flow (R) and steam flow or reboiler duty (QR). Numerical values of matrices A, B and D are given in [13].

As would be expected feed flowrate disturbance directly affects only the feed stage, i.e. state variables

h_5 and L_5 (liquid phase enthalpy and liquid flowrate at stage 5), and then propagates to all of the remaining state variables by dynamic coupling between each of the stages. It is then clear that as far as feed flowrate and enthalpy are concerned the rows of D corresponding to h_1 and h_{10} have zero entries. In other words feed-forward control action is not required for undisturbability of h_1 and h_{10} with respect to feed flowrate and feed enthalpy. Just like in the 11^{th} order model, the row of B corresponding to state variable h_{10} has two zero elements. Consequently, the partition B_1 is singular. However, the effect of control directly reaches h_9 and since there exists a signal flow path from h_9 to h_{10}, the original control objective is modified to make h_1 and h_9 undisturbable. It was known that in carrying out such a strategy, the original control objective would be realised because the row of A corresponding to h_{10} is invariant under any feedback. The design procedure for undisturbability was then applied to the augmented system with $\tilde{\mathbf{x}}^T = [h_1, h_9, h_{10}, h_2, h_3, \ldots, h_8, L_1, L_2, \ldots, L_{10}]$. With a choice $\tilde{W}_1 = I_2$, $\tilde{W}_2 = 0$, and $\Lambda_1 = \text{diag}(-2.5, -16.0)$, eigenvalues of the matrix P were evaluated (cf. eqn. (3.57)). The eigenvalues of P were found to be in the left half of the complex plane. With the check for stability complete, $\tilde{K}$ was evaluated from the equation:

$$\tilde{K} = \tilde{B}_1^{-1}([\tilde{A}_1 \ \tilde{A}_2] - [\tilde{H}_1 \ \ 0]), \text{ with } \tilde{H}_1 = \Lambda_1 \tag{4.24}$$

Notice that the computation for a state feedback controller for a 20th order system only requires inversion of a 2x2 diagonal matrix and its multiplication by a sparse 2x20 matrix. The point to note is that the design procedure for undisturbability is very simple and easy to use, and as illustrated by the above example it can often be carried out without the aid of a computer, even for such large order systems. A setpoint controller based on results of Section 4.2 was also designed for this system. The final control law for the above system was found to be

$$R = 4.95\ h_9 + 2.10\ h_{10} + 16.28\ h_8 - 47.99\ h_1^{SP}$$

$$Q_R = -17.69\ h_1, -3.40\ h_2 + 3.12\ L_2 + 37.48\ h_{10}^{SP} \tag{4.25}$$

The response of h_1 and h_{10} to unit step disturbances in feed flowrate with the control law of eqn. (4.25) implemented is shown in Figure 4.22. The performance of the setpoint controller was also evaluated by application of unit setpoint changes in h_1 and h_{10}. The remarkably

large degree of non-interaction achieved with a simple asymptotic setpoint controller is also evident in Figure 4.23.

A remark on the practicality of the control scheme of eqn. (4.25) is in order here. State variables h_1, h_2, h_8, h_9 and h_{10} are directly proportional to the temperatures at each of corresponding stage and hence these are easily available for state feedback. State variable, L_2, represents the liquid flowrate from the bottom tray to the reboiler. This measurement is not available at the present time on the pilot scale distillation column. However, a reasonable estimate of L_2, can be obtained by writing a material balance equation around the reboiler, and assuming that the vapour flowrate is directly proportional to steam flowrate into the reboiler. It is also important to emphasize here, that by using the structural aspects of the results on undisturbability, the form and structure of the control scheme (such as in eqn. (4.24)) for this or a similar system could have been predicted with a knowledge of only the zero elements of matrices A, B, and D. Such a preliminary analysis prior to actual design and modelling can be useful in planning of sensor locations.

It should also be noted that the control scheme

shown in eqn. (4.25) requires only measurement of 6 state variables. The reason for this is that because of the sparse open-loop system matrix $\tilde{A}$, and the zero elements of $\tilde{H}_2$ as required for undisturbability, a large number of columns of the difference matrix $([\tilde{A}_1 \quad \tilde{A}_2] - [\tilde{H}_1 \quad 0])$ are zero (cf. Section 2.4). Thus from eqn. (4.24) it is easy to see that feedback of state variables, corresponding to zero columns of the difference matrix, is not required.

4.6 Conclusions

The design procedure for undisturbability developed in Chapter 3 was applied to the design of feedback and feedforward controllers for the computer controlled, pilot-plant evaporator, an 11th order model of an 8 tray binary distillation column, and a 20th order model of an 8 tray pilot scale binary distillation column. The problem of simultaneous disturbance localisation and asymptotic setpoint tracking has also been discussed here. The proposed solution to this problem is both practical and simple. The possibility of having an independent integral feedback control of an undisturbable variable was also investigated and such a possibility was shown to be non-existent.

Experimental results from the computer-controlled pilot-plant evaporator at the University of Alberta demonstrated that the use of design procedure: produced the expected undisturbability; provided better control than conventional single variable controllers; and the performance of the resulting controllers was at least comparable if not better than controllers designed using other modern multivariable control techniques such as optimal-quadratic control. Evaluation of the controllers for the two distillation column models by digital simulation also demonstrated the effectiveness of the design method in eliminating the effect of specific disturbances on outputs of interest. The computational requirements for the design procedure were shown to be only marginal even in its application to large order systems. Experimental evaluation and digital simulation of setpoint controllers designed to retain the disturbance localisation property and at the same time asymptotically track a certain class of inputs gave good results and confirmed the practicality of such an approach.

In general, the design procedure for undisturbability was found to be practical and convenient. It was easy to use, gave considerable insight into system performance, placed the closed-loop eigenvalues and

eigenvectors in the desired locations, produced practical controllers with reasonable gains, and performed well experimentally. To the best of the author's knowledge, this is the first reported experimental application of this design approach.

TABLE 4.1

Summary of experimental runs performed on the evaporator

Run No.	Controller	Eigenvalues and other design details	Experimental run conditions
DL1/P/1 Figure 4.2	$K_{d\ell 1} = \begin{bmatrix} 2.70 & 0 & -9.69 \\ 3.23 & 0 & 3.88 \\ 4.29 & 13.05 & 17.17 \end{bmatrix}$	Λ = diag(.65,.47,.28) MO = 3; FB*	$\mp$ 30% step in CF
Figure 4.5	as above	as above	$\pm$ 20% step in F
DL2/P/1 Figure 4.6	$K_{d\ell 2} = \begin{bmatrix} 3.18 & 0 & -8.77 \\ 3.80 & 0 & 3.51 \\ 5.03 & 14.11 & 15.53 \end{bmatrix}$	Λ = diag(.59,.43,.34) MO = 3; FB	$\pm$ 20% step in F
DL1/P+FF/2 Figure 4.4	Feedback, $K_{d\ell 1}$, as above plus feedforward K^{FF1} such that: $\begin{bmatrix} S \\ B1 \\ B2 \end{bmatrix} = \begin{bmatrix} .31 \\ -.12 \\ -.55 \end{bmatrix} F$	Λ = Run DL1/P/1 FF+FB to make C2 Undisturbable by F MO = 3; FB+FF	$\pm$ 20% step in F

* For explanation of Run No. and other notation see section on Notation, page 250.

TABLE 4.1 (Continued)

DL1/P+SP/3 Figure 4.17	Feedback, $K_{d\ell 1}$, as above plus setpoint controller: $K^{SP1} = \begin{bmatrix} -2.7 & 0 & 10.26 \\ -3.23 & 0 & -4.11 \\ -4.29 & -13.05 & -18.18 \end{bmatrix}$	Λ = Run DL1/P/1 K_{SP} designed to achieve steady-state input/output decoupling MO = 3; FB+SP	± 10% step in set-point of C2
ML/P/1 Figure 4.7	$K_{m\ell} = \begin{bmatrix} 0 & 0 & -4.98 \\ 2.53 & 0 & 0 \\ 0 & 6.5 & 0 \end{bmatrix}$	Λ = diag(0.74, .75±j,.13) Proportional FB constants tuned experimentally FB	± 20% step in F
Figure 4.3	as above	as above	∓ 30% step in CF
ML/P+FF/2 Figure 4.8	Feedback, $K_{m\ell}$, as above plus feedforward, K^{FF2} such that: $\begin{bmatrix} S \\ B1 \\ B2 \end{bmatrix} = \begin{bmatrix} 1.24 \\ 0.98 \\ 1.0 \end{bmatrix} F$	Λ = Run ML/P/1 Proportional FB constants tuned experimentally FB+FF	± 20% step in F

TABLE 4.1 (Continued)

OPT/P/1 Figure 4.9	$K_{opt1} = \begin{bmatrix} 4.9 & -0.4 & -11.92 \\ 5.78 & -1.6 & 4.43 \\ 4.09 & 9.69 & 9.36 \end{bmatrix}$	Λ = diag(.08,.30,.62) Q = diag(10,10,100) R = diag(.05,.05,.05) MO = 3; FB	± 20% step in F
Figure 4.10	as above	as above	∓ 30% step in CF
OPT/P+SP/2 Figure 4.19	Feedback, K_{opt1}, as above plus setpoint controller: $K^{SP2} = \begin{bmatrix} -4.9 & 0.4 & 12.47 \\ -5.78 & 1.6 & -4.65 \\ -4.09 & -9.69 & -10.35 \end{bmatrix}$	Λ = Run OPT/P/1 Q = Run OPT/P/1 R = Run OPT/P/1 MO = 3; FB+SP	± 10% step in setpoint of C2
DL5/P/1 Figure 4.11	$K_{d\ell 5} = \begin{bmatrix} 9.26 & -1.98 & -7.65 & 0 & -34.13 \\ 2.94 & 0.28 & 0 & 0 & 4.97 \\ 2.99 & 1.23 & 0.01 & 13.05 & 21.99 \end{bmatrix}$	Λ = diag(.28, .47, .65, -.76,.9) MO = 5; FB	± 20% step in F

TABLE 4.1 (Continued)

DL5/P+FF/2 Figure 4.12	Feedback, $K_{d\ell 5}$, as above plus feedforward, K^{FF3} such that $\begin{bmatrix} S \\ B1 \\ B2 \end{bmatrix} = \begin{bmatrix} .89 \\ .24 \\ .22 \end{bmatrix} F$	Λ = Run DL5/P/1 FF+FB to make C2 and W2 undisturbable by F MO = 5; FB+FF	± 20% step in F
DL5/P+FF/3 Figure 4.13	Feedback, $K_{d\ell 5}$, as above plus feedforward, K^{FF4} such that $\begin{bmatrix} s \\ B1 \\ B2 \end{bmatrix} = \begin{bmatrix} 3.22 \\ 0.98 \\ 0.98 \end{bmatrix} F$	Λ = Run DL5/P/; FF+FB to make C2, W2 and W1 undisturbable by F MO = 5; FB+FF	± 20% step in F
DL5/P+SP/4 Figure 4.18	Feedback, $K_{d\ell 5}$, as above plus setpoint controller: $K^{SP3} = \begin{bmatrix} -9.26 & 0 & 36.11 \\ -2.94 & 0 & -5.26 \\ -2.99 & -13.05 & -23.26 \end{bmatrix}$	Λ = Run DL5/P/1 K^{SP3} designed to achieve steady-state input/output decoupling MO = 5; FB+SP	± 10% step in setpoint of C2
OPT/P/5 Figure 4.14	$K_{opt5} = \begin{bmatrix} 6.51 & -1.23 & -3.24 & -0.09 & -13.14 \\ 3.83 & 0.37 & 0.69 & -1.38 & 9.71 \\ 3.06 & 1.06 & 0.18 & 9.75 & 11.56 \end{bmatrix}$	$\Lambda = (.90, .61, .33 \pm j, .12, .13)$ (Run no. 50PT, Table 6.7, Wilson [15]) MO = 5; FB	± 20% step in F

TABLE 4.1 (Continued)

MC/p/1 Figure 4.16	$G_{19} = \begin{matrix} 3.11 & -1.16 & -9.40 \\ 5.92 & -2.97 & 4.29 \\ 9.06 & 2.85 & 18.89 \end{matrix}$	Λ = diag(.43,61, .64±j.34,.9) (Modal controller G_{19} from Topaloglu [10]) MO = 5; FB(output)	± 20% step in F
EA/P/1 Figure 4.15	$K_{16} = \begin{bmatrix} 10.48 & -6.83 & -5.20 & .04 & -16.79 \\ 8.94 & 13.69 & 0.83 & .05 & 3.98 \\ 14.23 & 14.81 & -0.46 & 24.68 & 24.97 \end{bmatrix}$	Λ = diag(0,0±j0, .27,.90) Eigenvalue assignment by state feedback. Run no. 16 from Park [9]) MO = 5; FB	± 20% step in F

TABLE 4.2

Numerical values of the closed-loop system matrix, $\tilde{H}$, and the disturbance matrix, $\tilde{D}$, for the 11[th] order distillation column model.

$$\tilde{H} = \left[\begin{array}{ccc:c|ccccccc}
-0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0.0043 & 0 & 0 & -0.014 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-0.272 & -0.206 & -0.008 & -0.013 & -0.019 & 0.006 & 0 & 0 & 0 & 0 & -0.021 \\
-0.235 & -0.239 & -0.009 & -0.009 & 0.006 & -0.016 & -0.011 & 0 & 0 & 0 & -0.024 \\
-0.188 & -0.272 & -0.010 & -0.008 & -0.003 & 0.010 & -0.031 & +0.015 & 0 & 0 & -0.027 \\
-0.188 & -0.272 & -0.010 & -0.008 & -0.003 & 0 & 0.020 & -0.035 & 0.022 & 0 & -0.027 \\
-0.162 & -0.290 & -0.009 & -0.007 & -0.003 & 0 & 0 & 0.020 & -0.042 & 0.028 & -0.029 \\
-0.132 & -0.311 & -0.008 & -0.005 & -0.002 & 0 & 0 & 0 & 0.020 & -0.048 & 0.006 \\
0.098 & -0.256 & 0.005 & 0.004 & 0.002 & 0 & 0 & 0 & 0 & 0.020 & -0.084
\end{array}\right]
\qquad
\tilde{D} = \left[\begin{array}{c}
0 \\ 0 \\ 0 \\ \hdashline 0 \\ \hline 0 \\ 0 \\ 0.011 \\ 0 \\ 0 \\ 0 \\ 0
\end{array}\right]$$

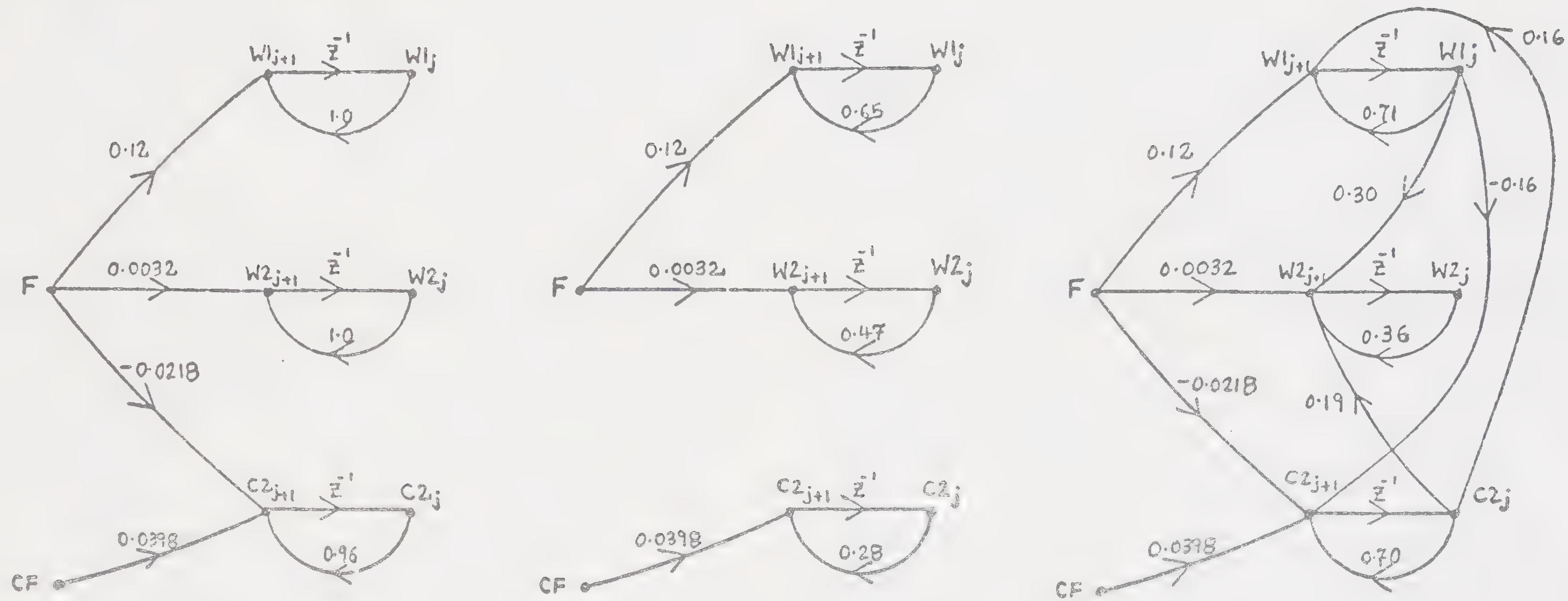


Figure 4.1: Signal flow graphs of the evaporator system under open-loop conditions (left); with a disturbance localisation controller (centre); and with conventional multiloop control (right).

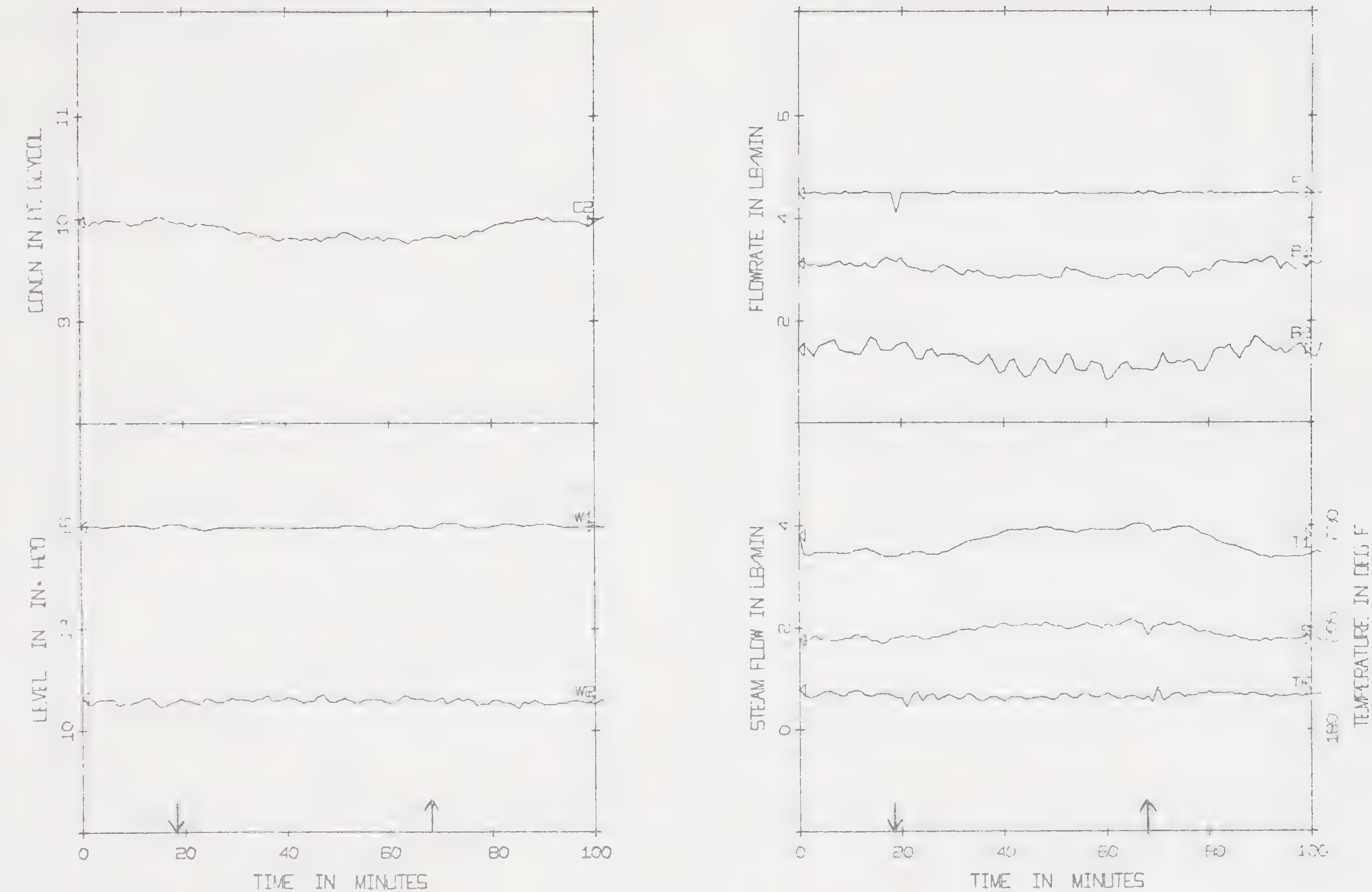


Figure 4.2: Experimental proportional feedback control (DL1/P/1)* ($K_{d\ell 1}$ (Table 4.1); MO = 3; FB; $\mp$ 30% CF (at time indicated by arrows on t axis))*.

* For explanation of computer plot notation, see section on notation, page 250.

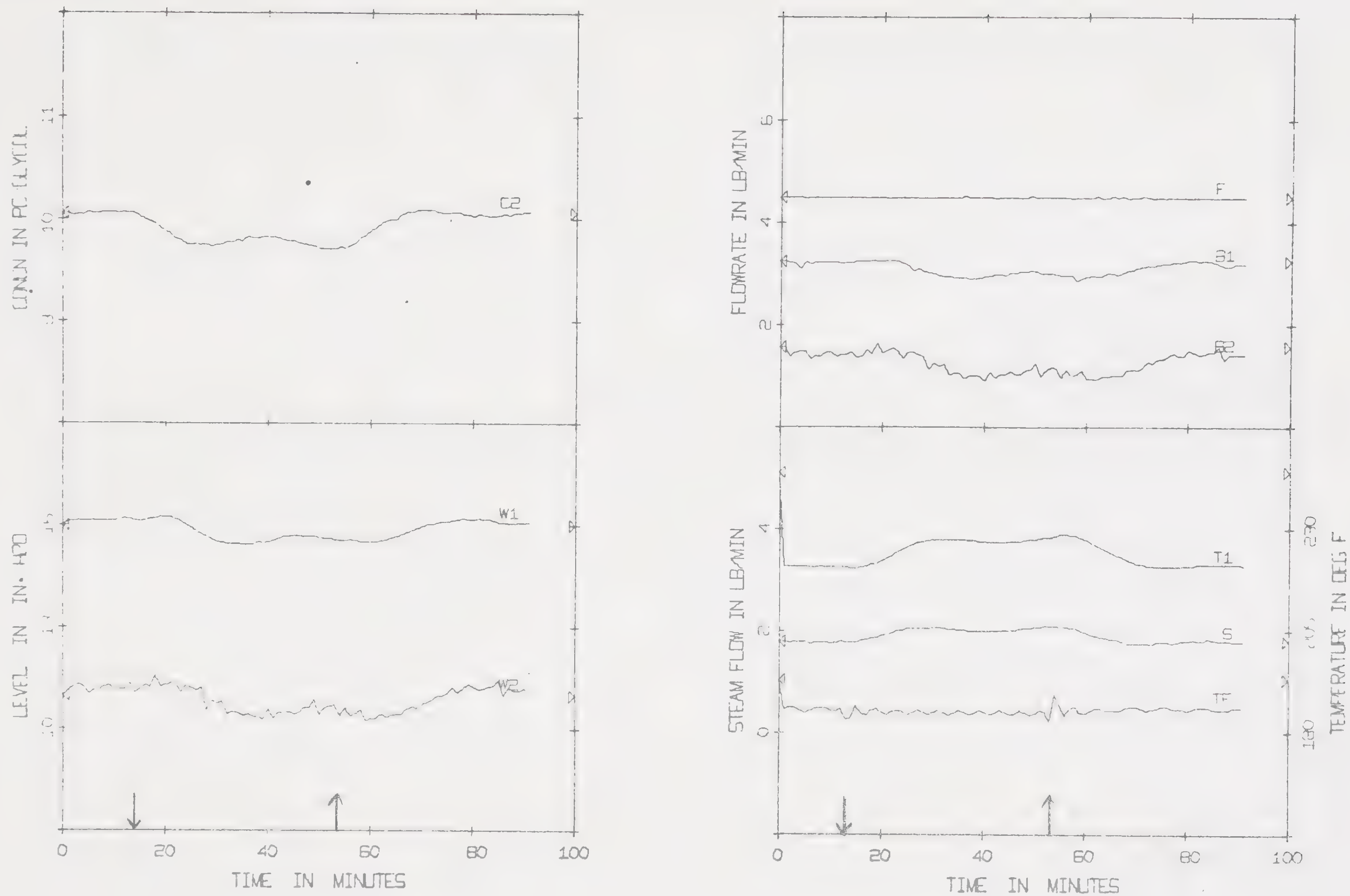


Figure 4.3: Experimental proportional feedback control (ML/P/1) ($K_{m\ell}$(Table 4.1); MO = 3; FB; $\mp$ 30% CF)

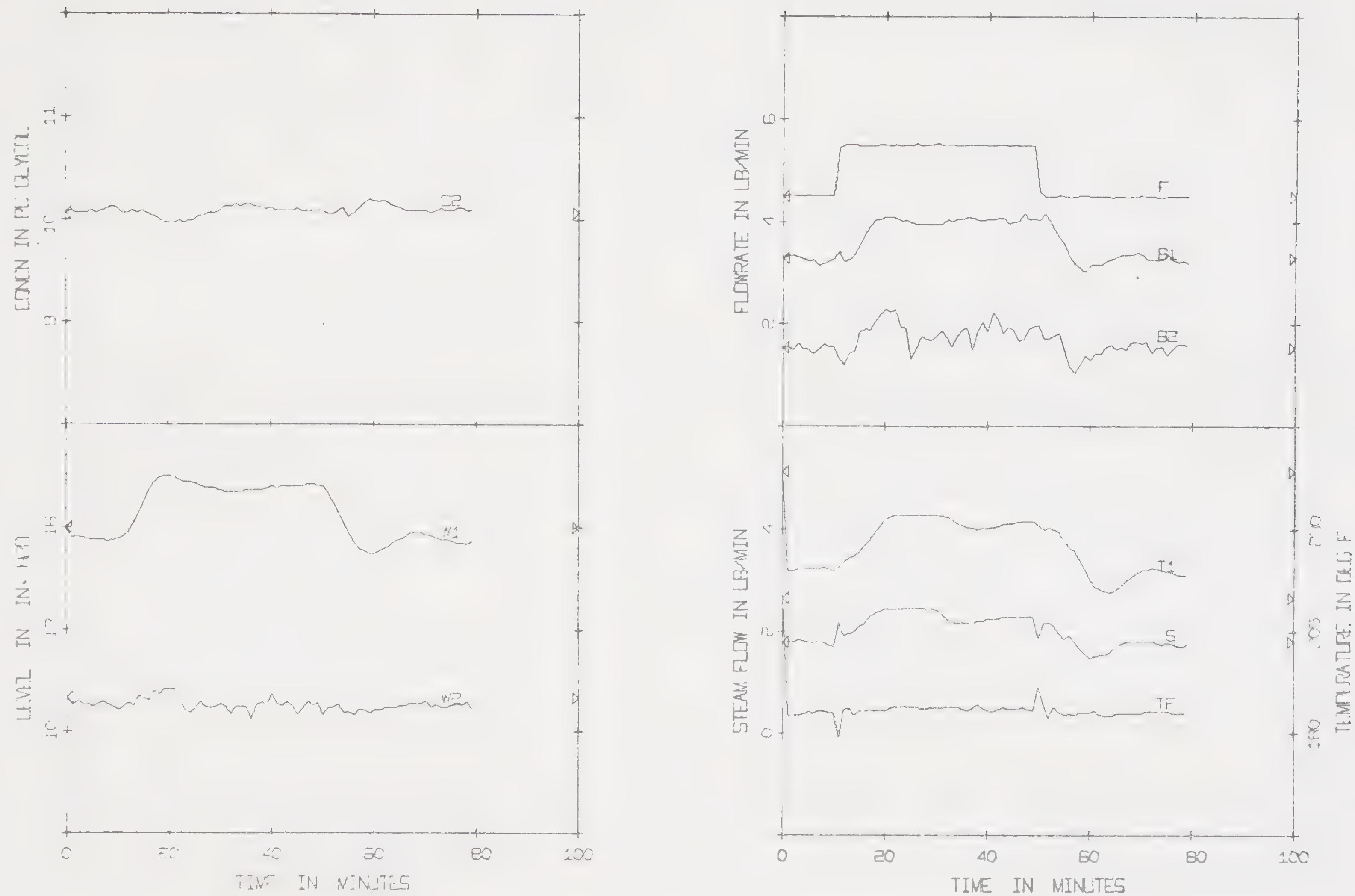


Figure 4.4: Experimental proportional feedback plus feedforward control (DL1/P + FF/2) ($K_{d\ell 1}$ + K^{FF1} (Table 4.1); MO = 3; FB + FF; ± 20%F).

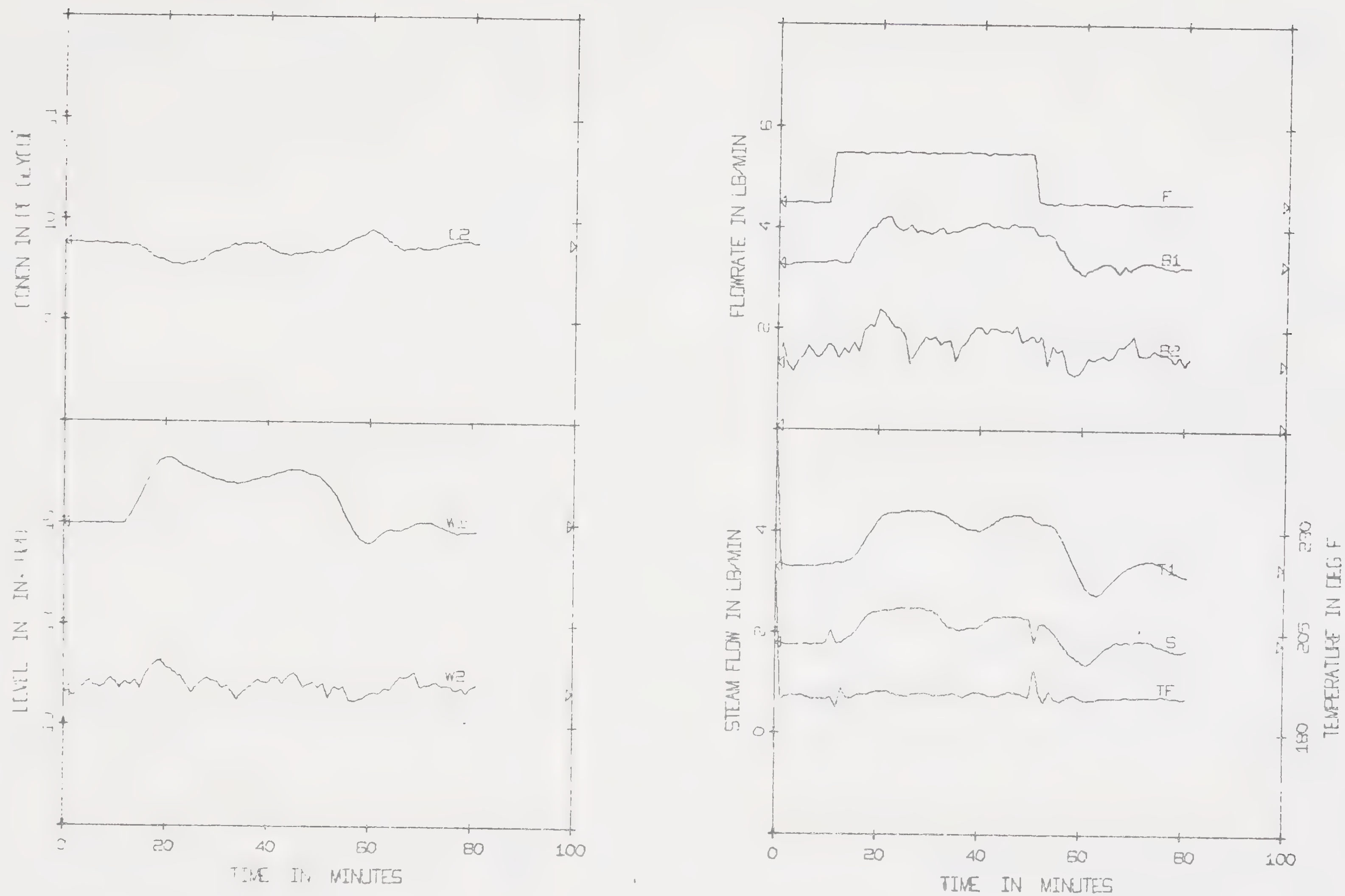


Figure 4.5: Experimental proportional feedback control (DL1/P/1) ($K_{d\ell 1}$(Table 4.1); MO = 3; FB; ± 20%F).

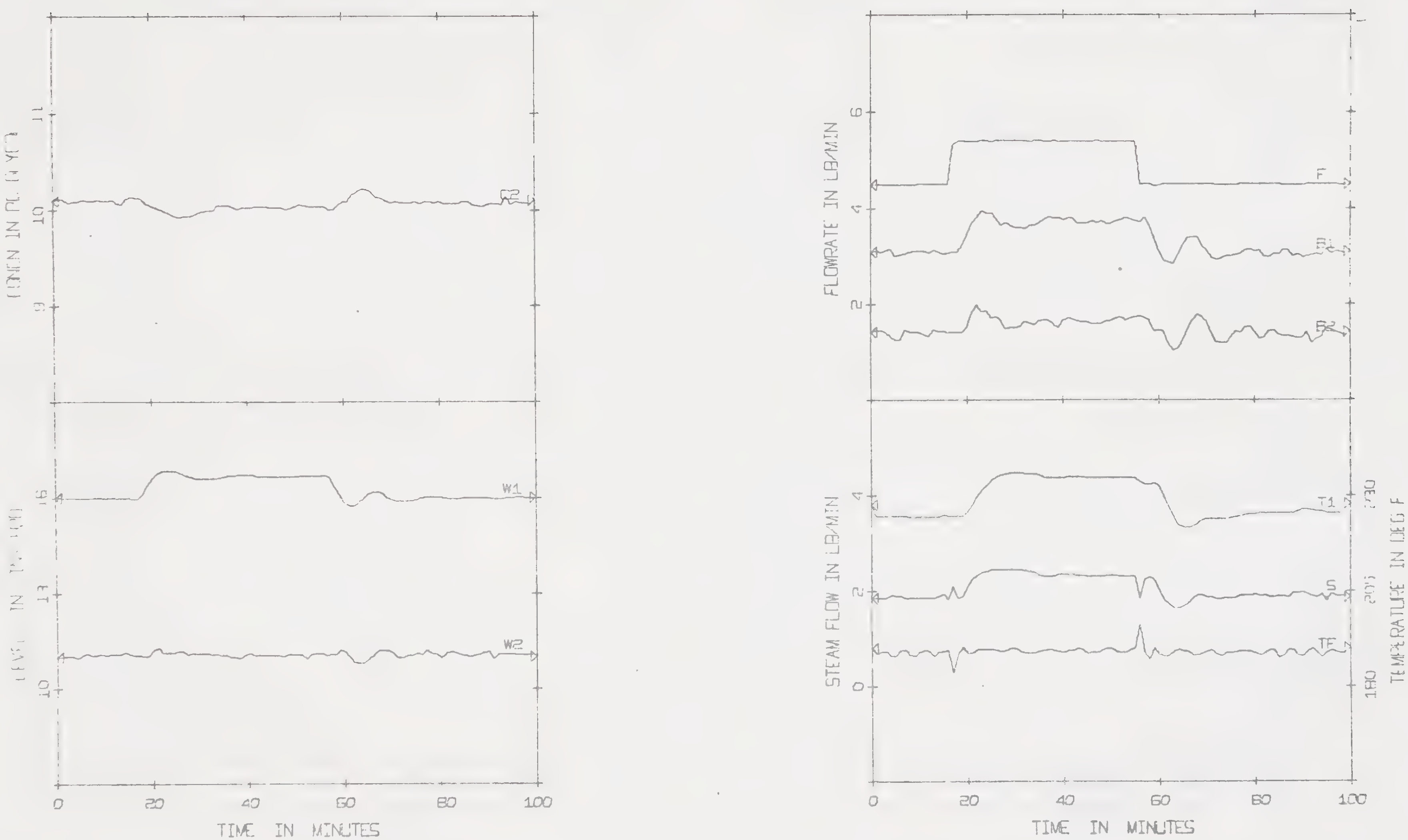


Figure 4.6: Experimental proportional feedback control (DL2/P/1) ($K_{d\ell 2}$(Table 4.1); MO = 3; FB; ± 20% F).

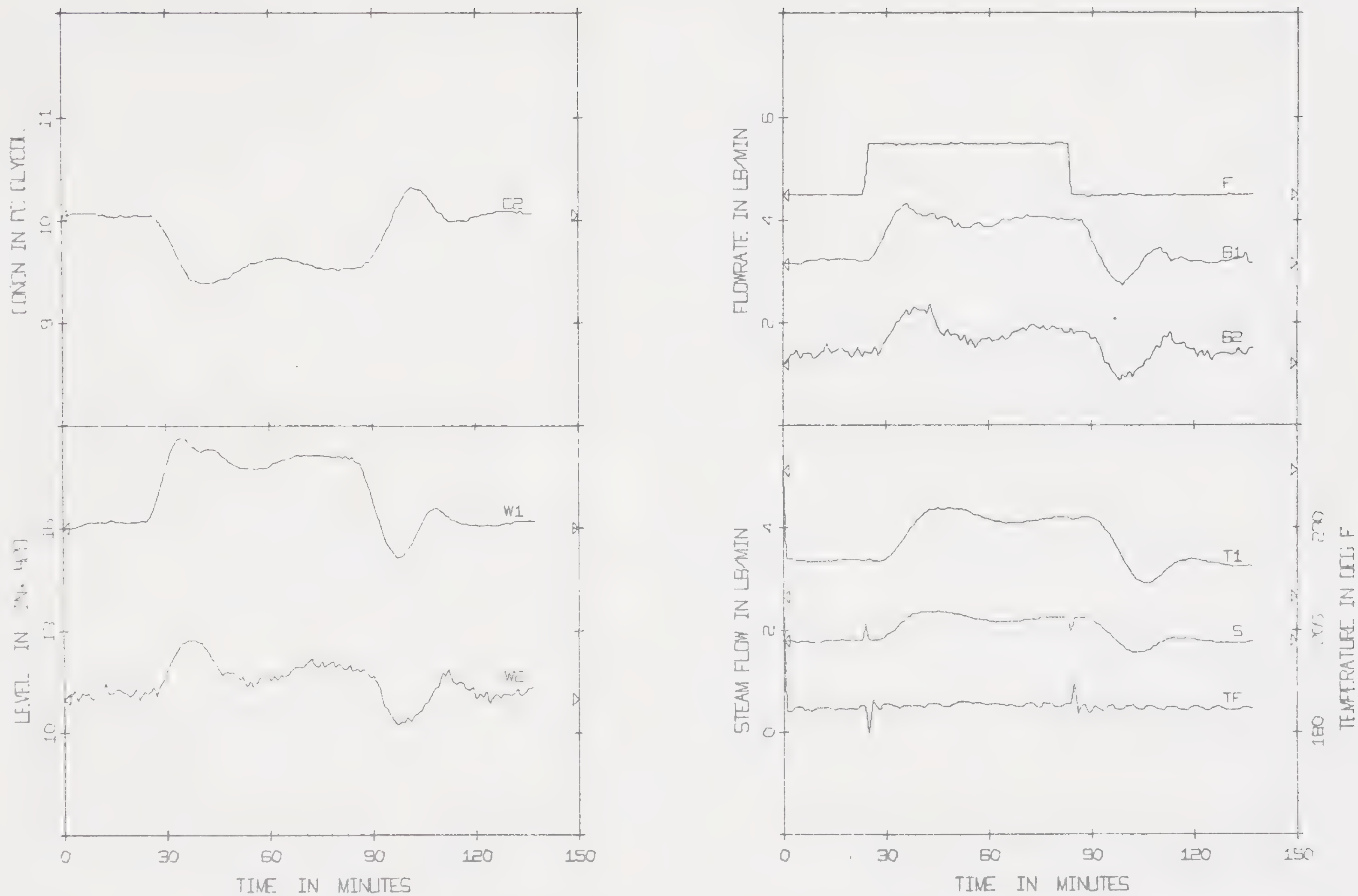


Figure 4.7: Experimental proportional feedback control (ML/P/1) ($K_{m\ell}$(Table 4.1); MO = 3; FB; ± 20%F).

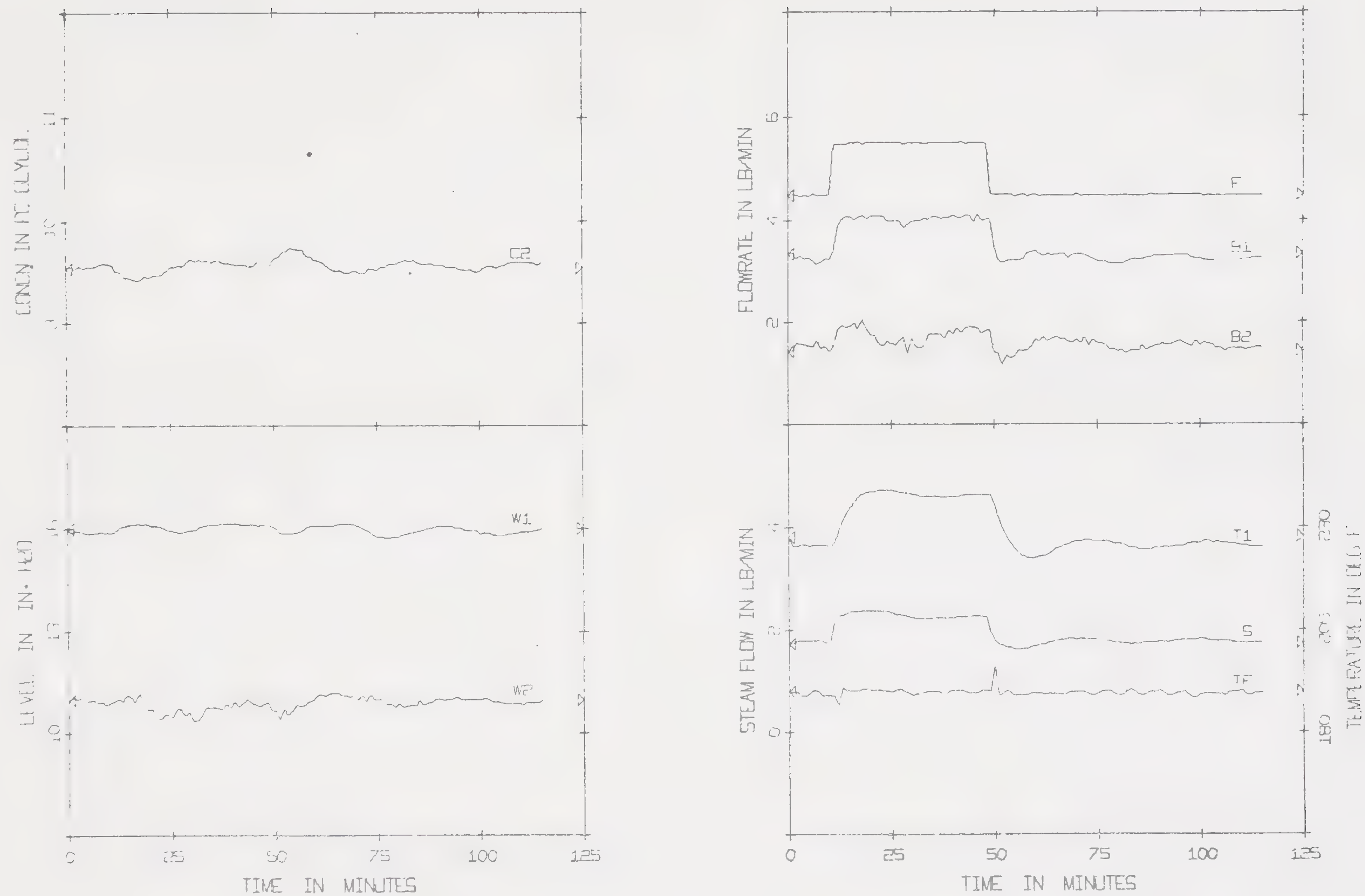


Figure 4.8: Experimental proportional feedback plus feedforward control (ML/P + FF/2) ($K_{m\ell}$ + K^{FF2} (Table 4.1); FB + FF; ± 20%F).

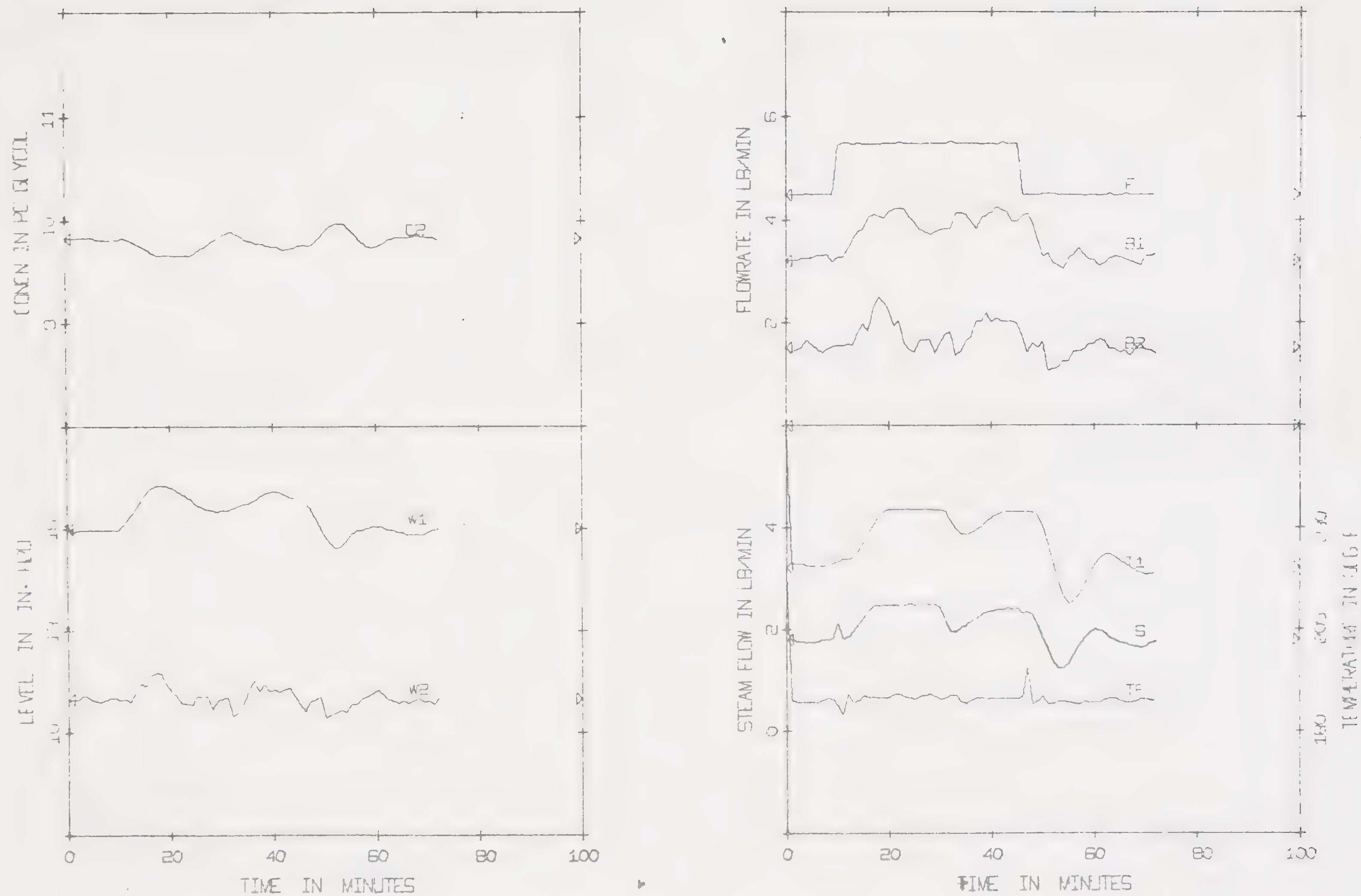


Figure 4.9: Experimental porportional feedback control (OPT/P/1) (K_{opt1} (Table 4.1); MO = 3, FB; ± 20% F).

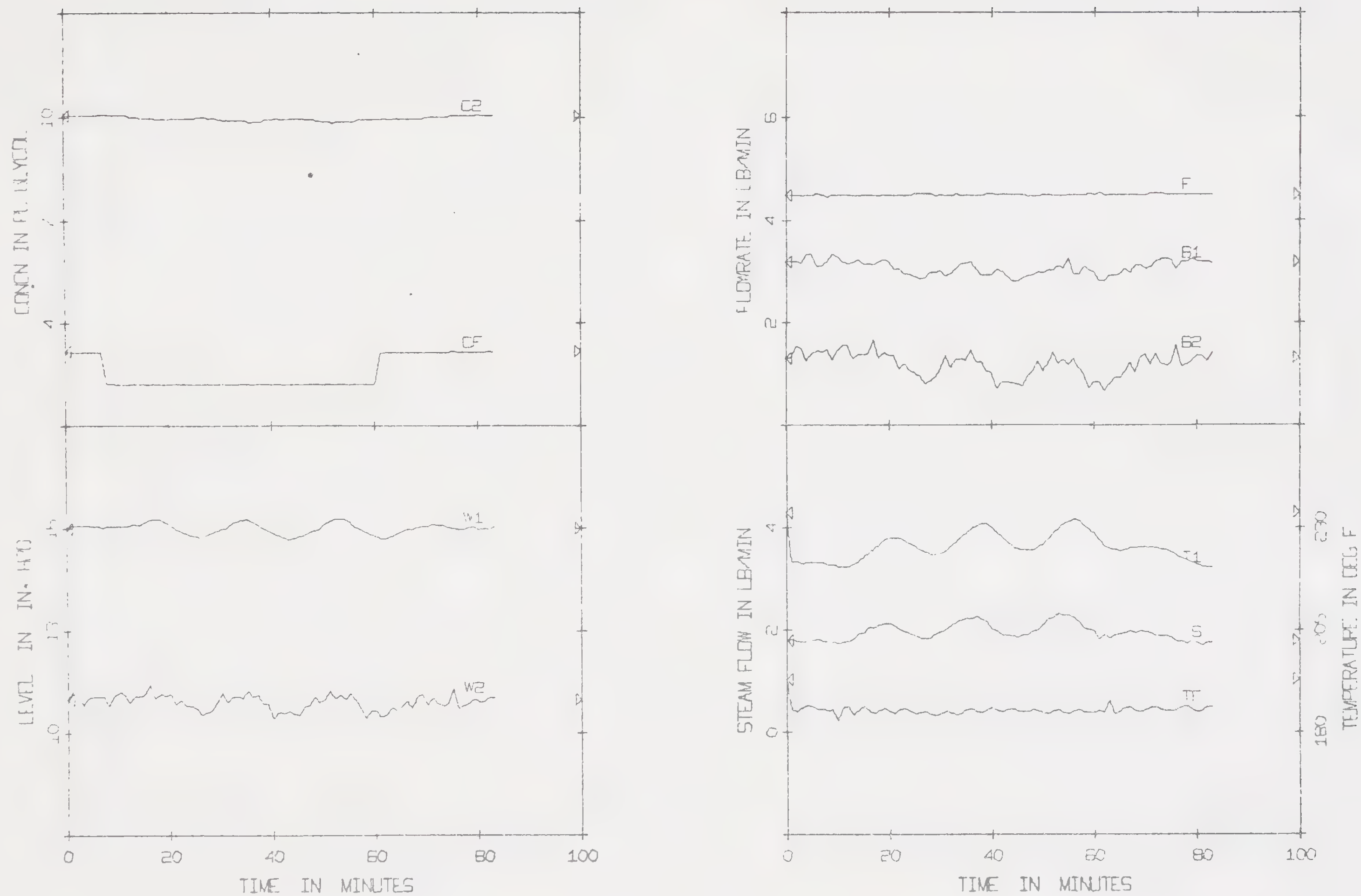


Figure 4.10: Experimental proportional feedback control (OPT/P/1) (K_{opt1} (Table 4.1); MO = 3; FB; $\mp$ 30% CF).

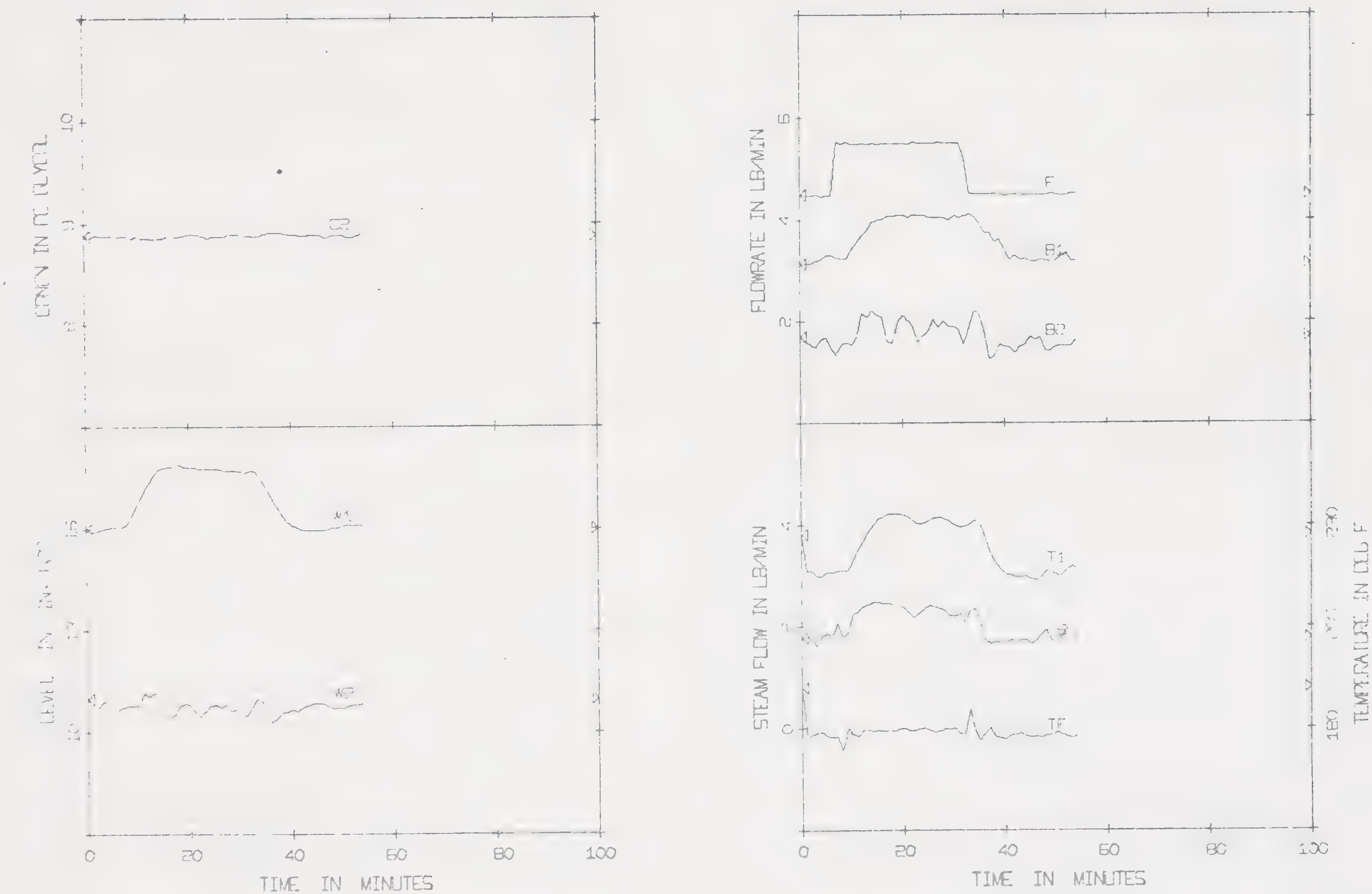


Figure 4.11: Experimental proportional feedback control (DL5/P/1) ($K_{d\ell 5}$ (Table 4.1); MO = 5; FB; ± 20%F).

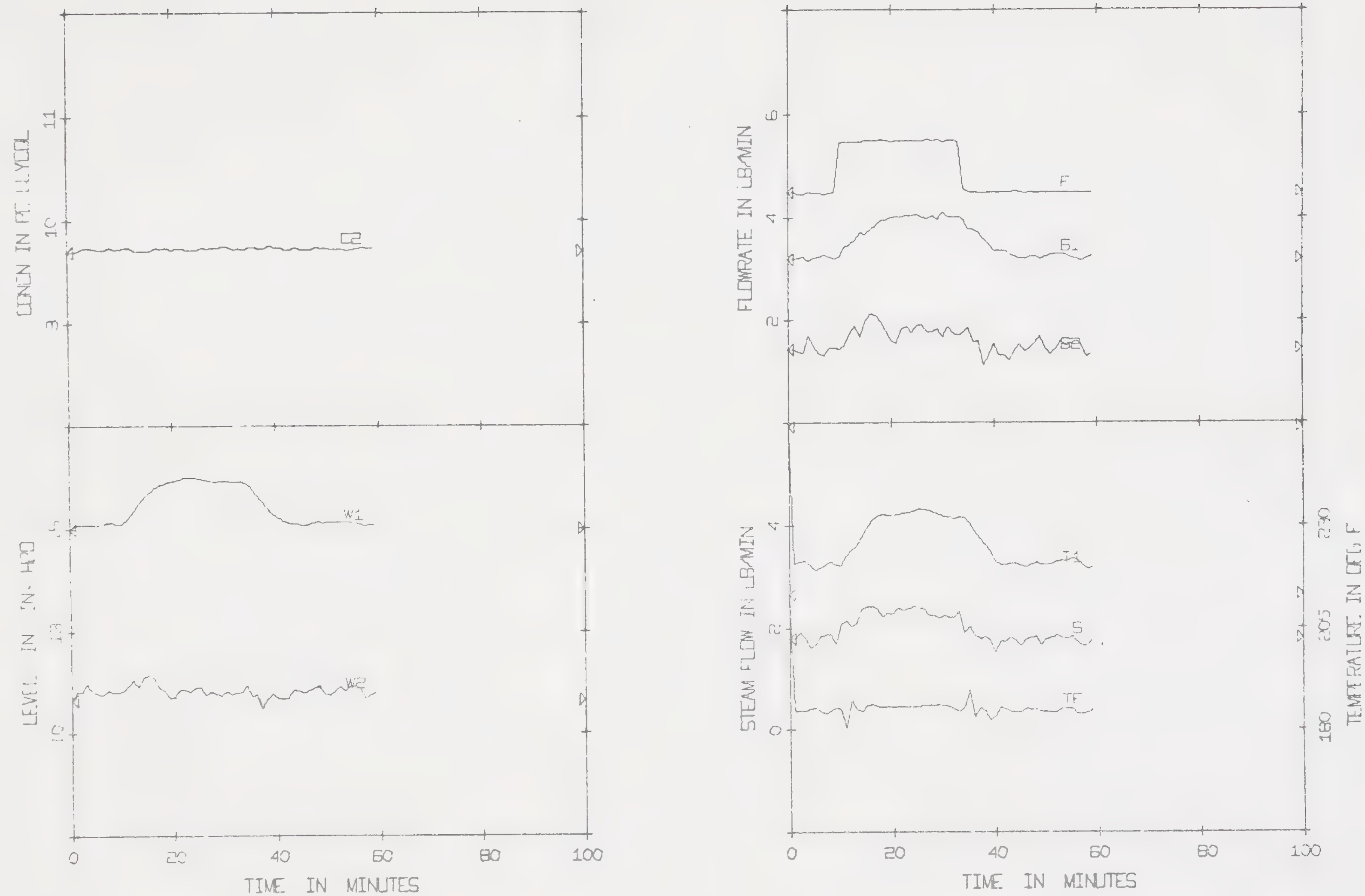


Figure 4.12: Experimental proportional feedback plus feedforward control (DL5/P + FF/2) ($K_{d\ell 5}$ + K^{FF3} (Table 4.1); MO = 5; FB + FF; ± 20%F).

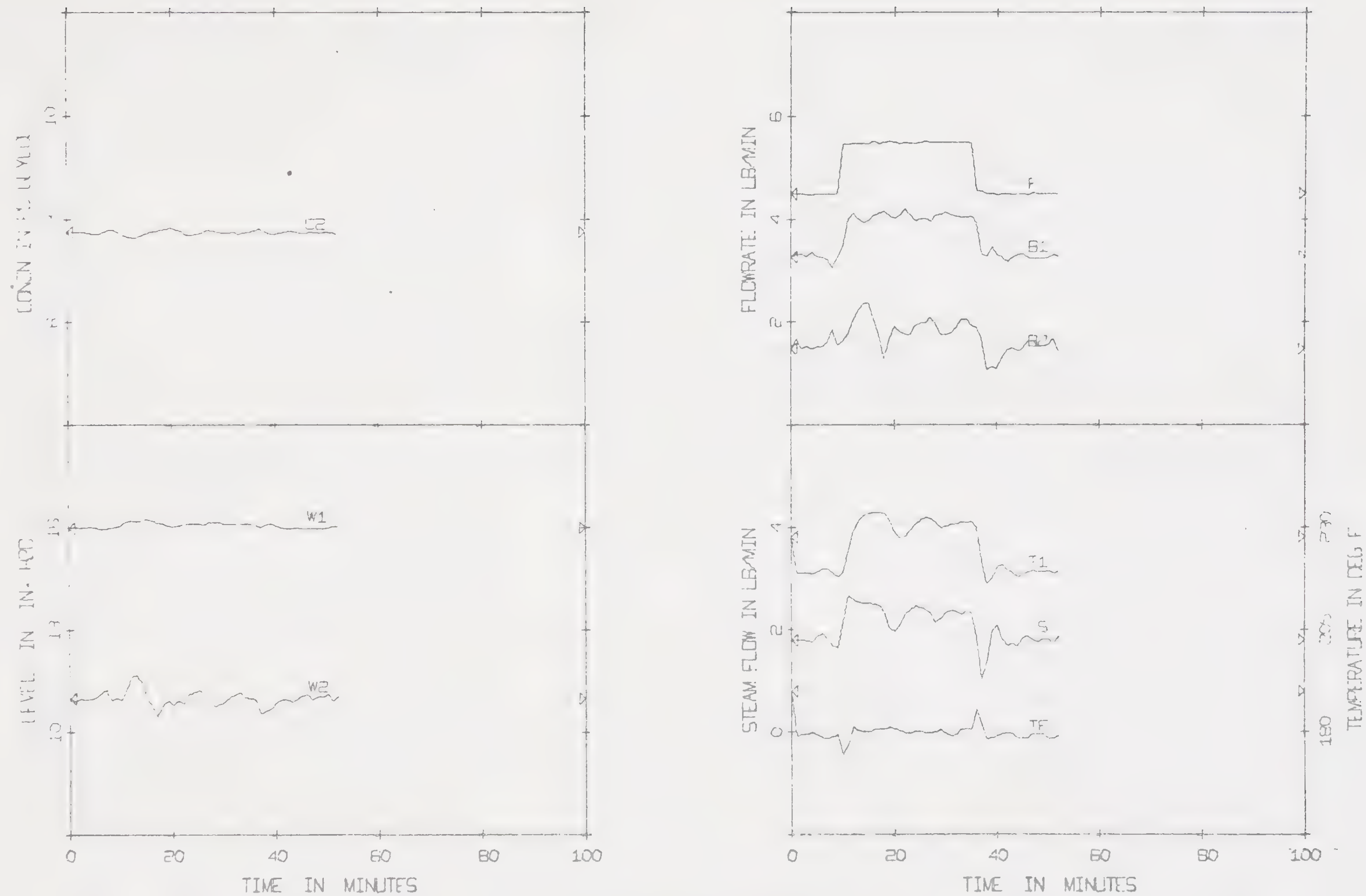


Figure 4.13: Experimental proportional feeback plus feedforward control (DL5/P + FF/3) ($K_{d\ell 5}$ + K^{FF4} (Table 4.1); MO = 5; FB + FF; ± 20%F).

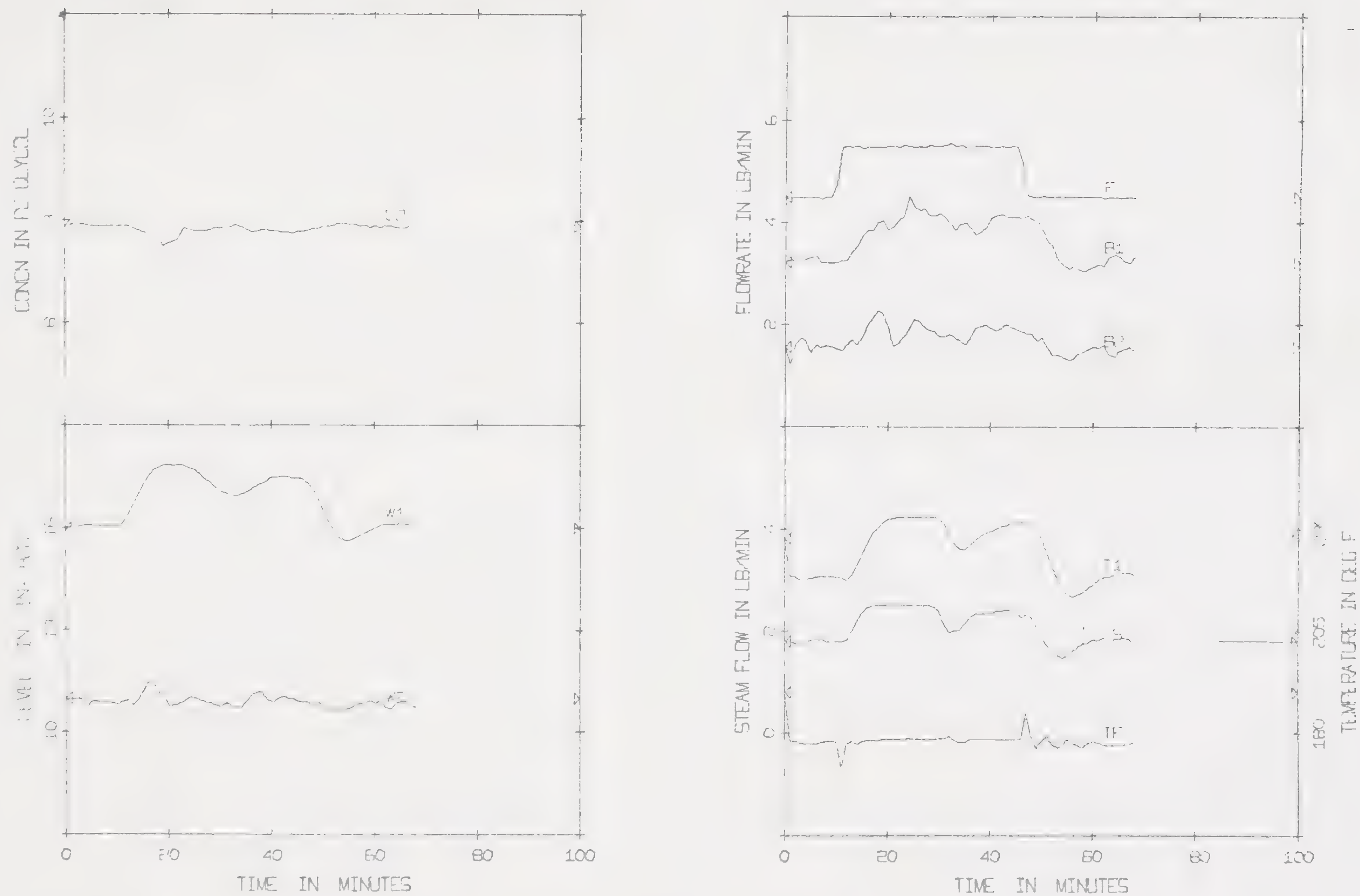


Figure 4.14: Experimental proportional feedback control (OPT/P/5) (K_{opt5}(Table 4.1); MO = 5; FB; ± 20%F).

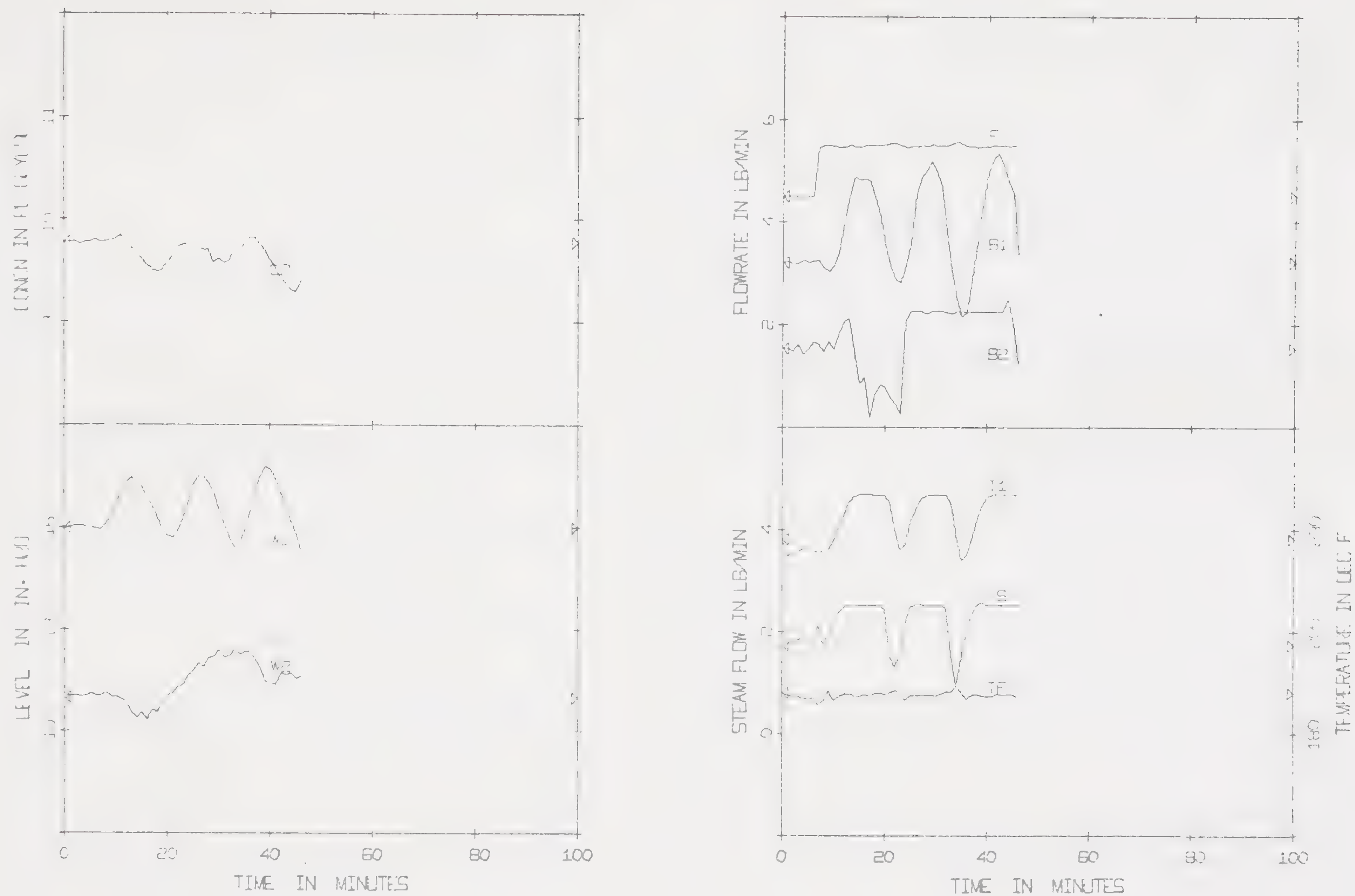


Figure 4.15: Experimental proportional feedback control (EA/P/1) (K_{16} (Table 4.1); MO = 5; FB; ± 20% F).

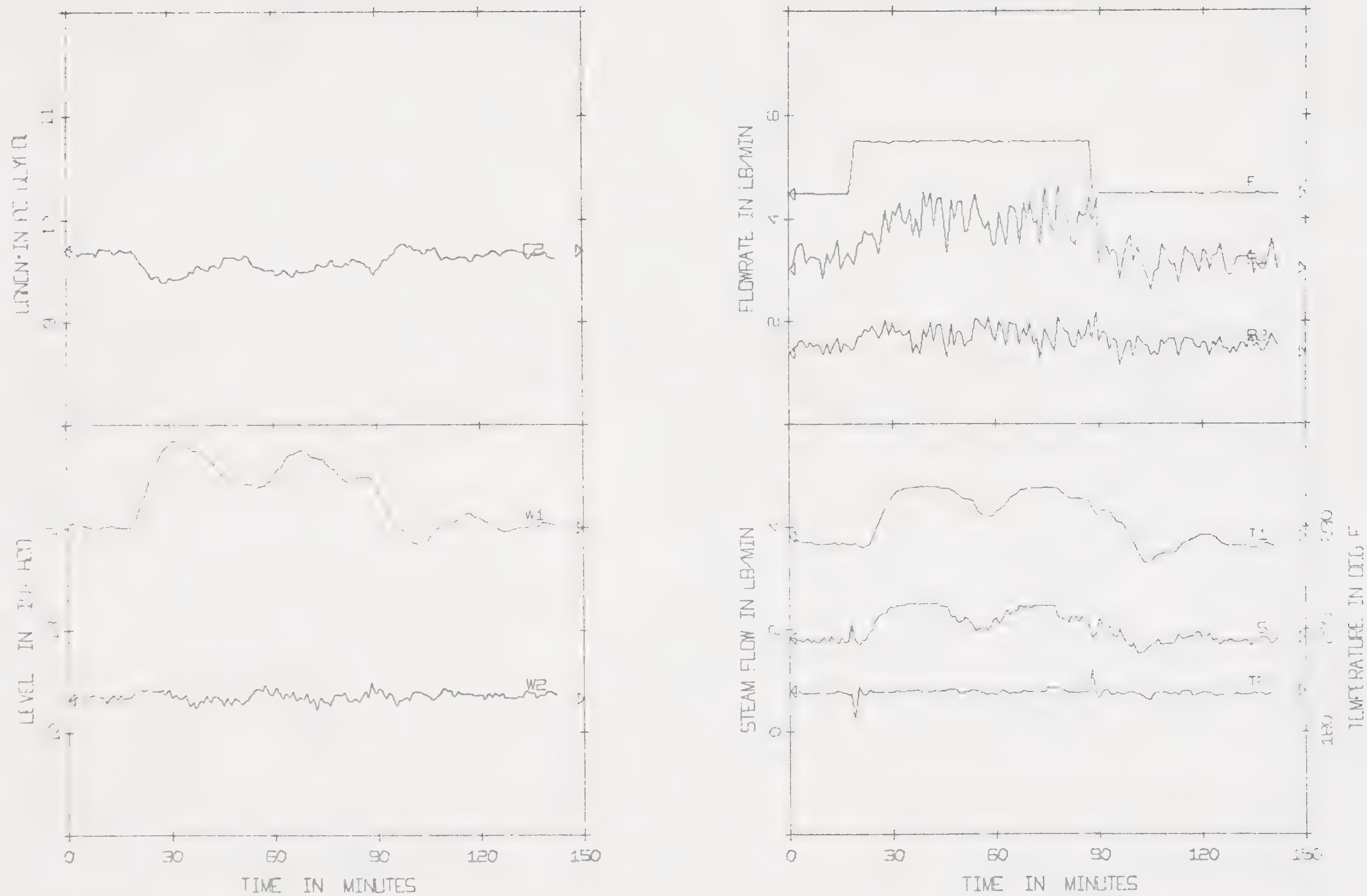


Figure 4.16: Experimental proportional output feedback control (MC/P/1) (G_{19}(Table 4.1); MO = 5; (output)FB; ± 20%F).

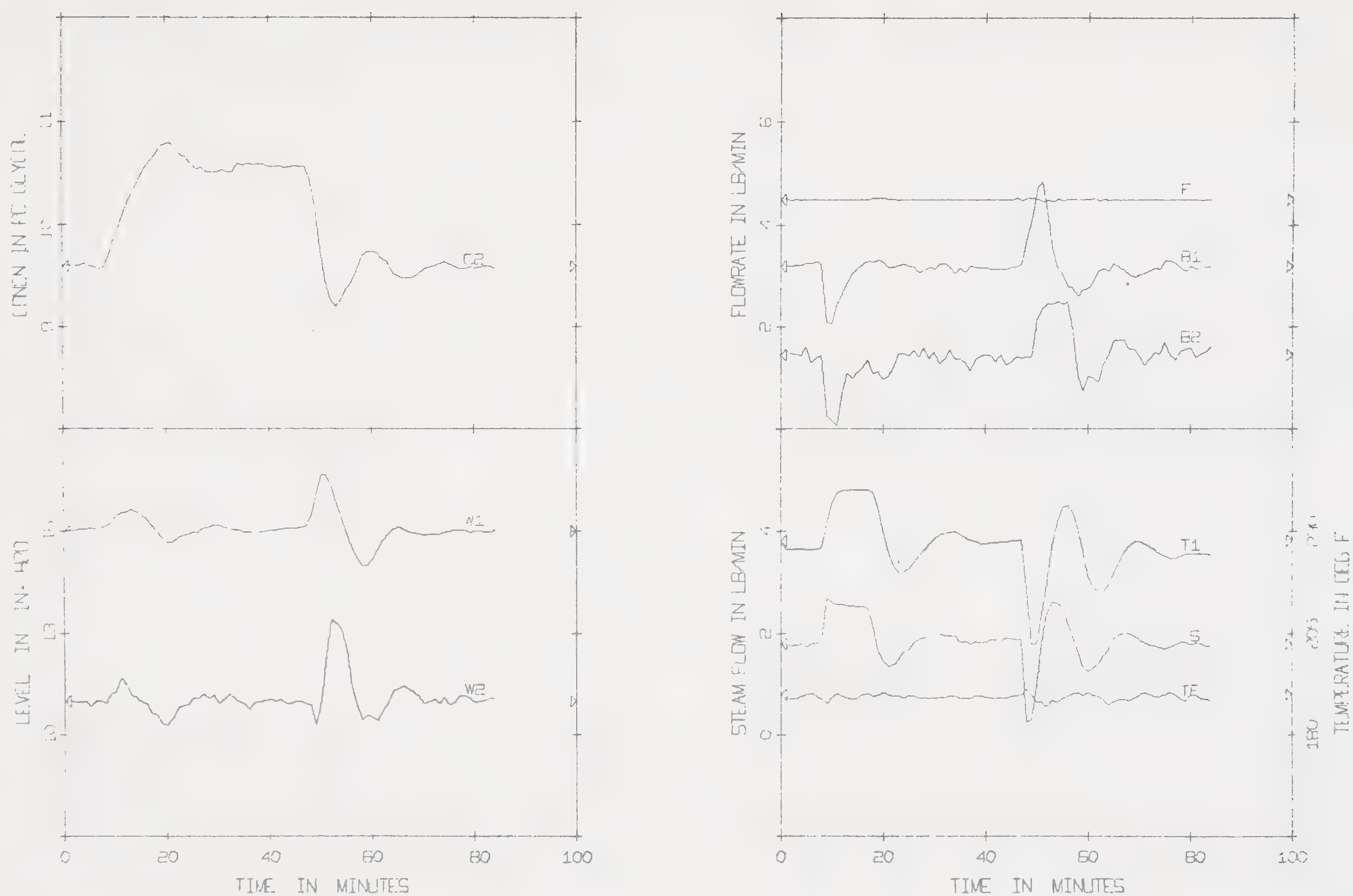


Figure 4.17: Experimental setpoint change (DL1/P + SP/3) ($K_{d\ell 1} + K^{SP1}$ (Table 4.1); MO = 3; FB + SP, ± 10% C2).

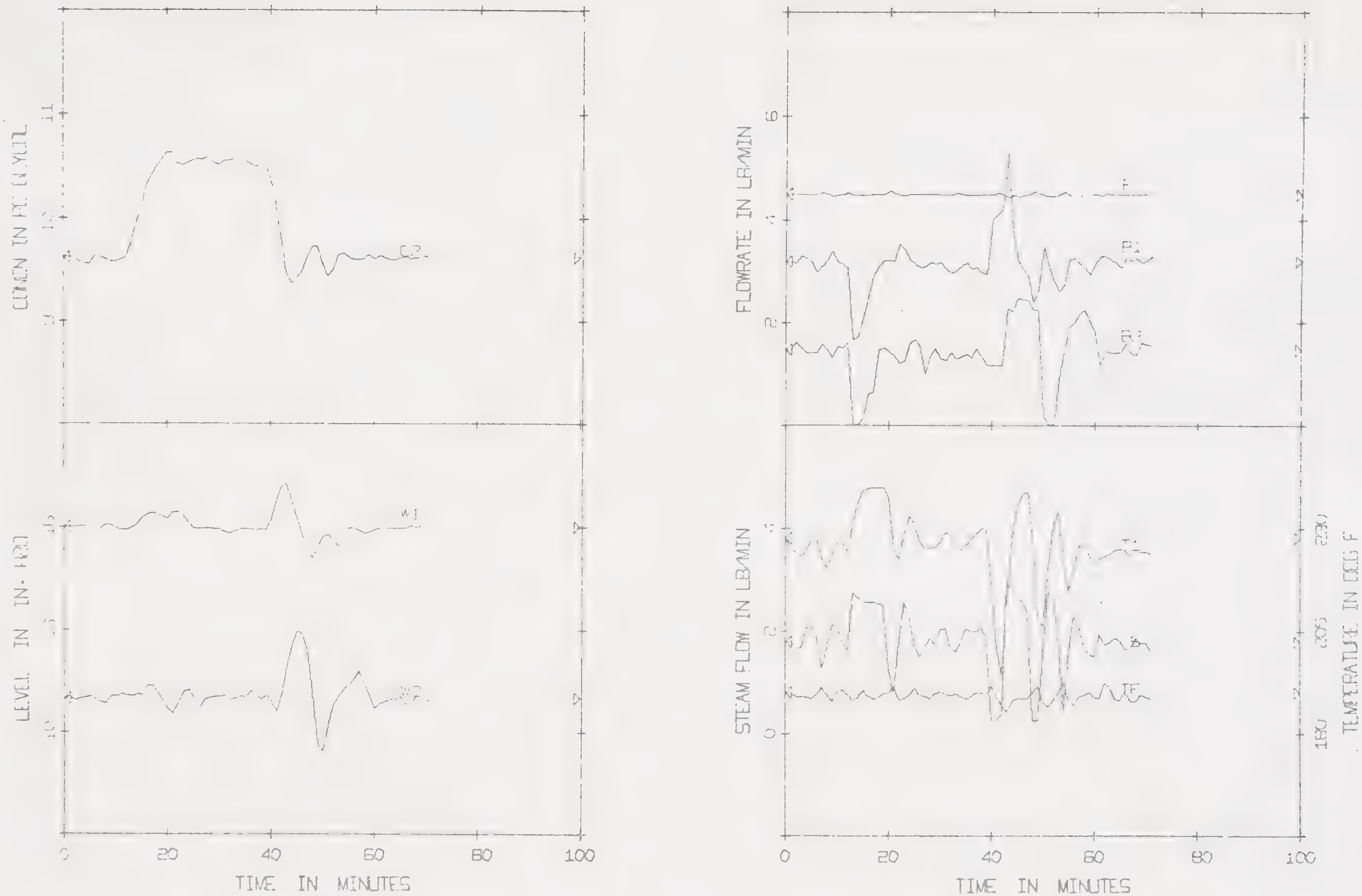


Figure 4.18: Experimental setpoint change (DL5/P + SP/4) ($K_{d\ell 5} + K^{SP3}$ (Table 4.1); MO = 5; FB + SP; ± 10% C2).

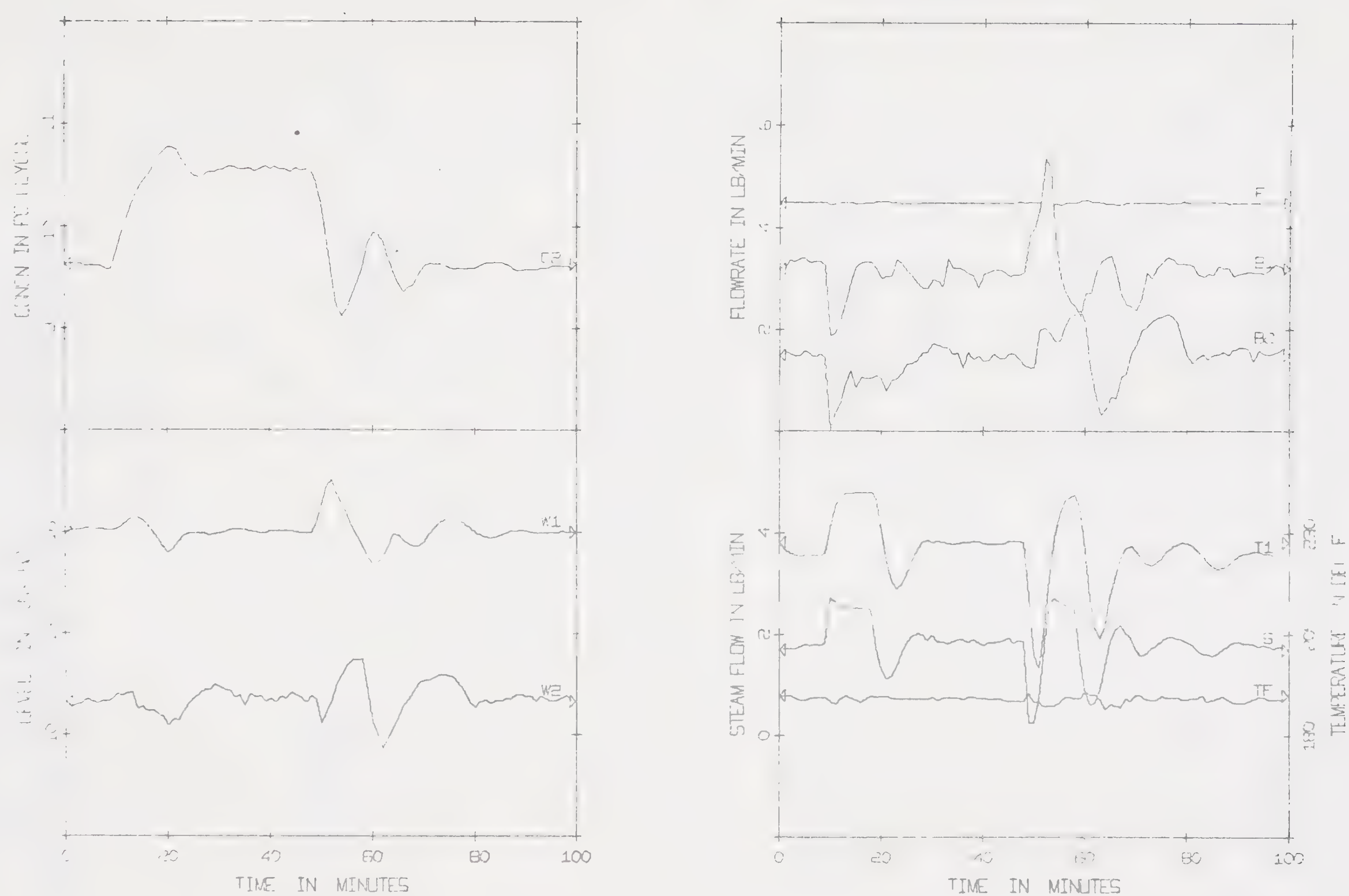


Figure 4.19: Experimental setpoint change (OPT/P + SP/2)
(K_{opt1} + K^{SP2} (Table 4.1); MO = 3; FB + SP; ± 10% C2).

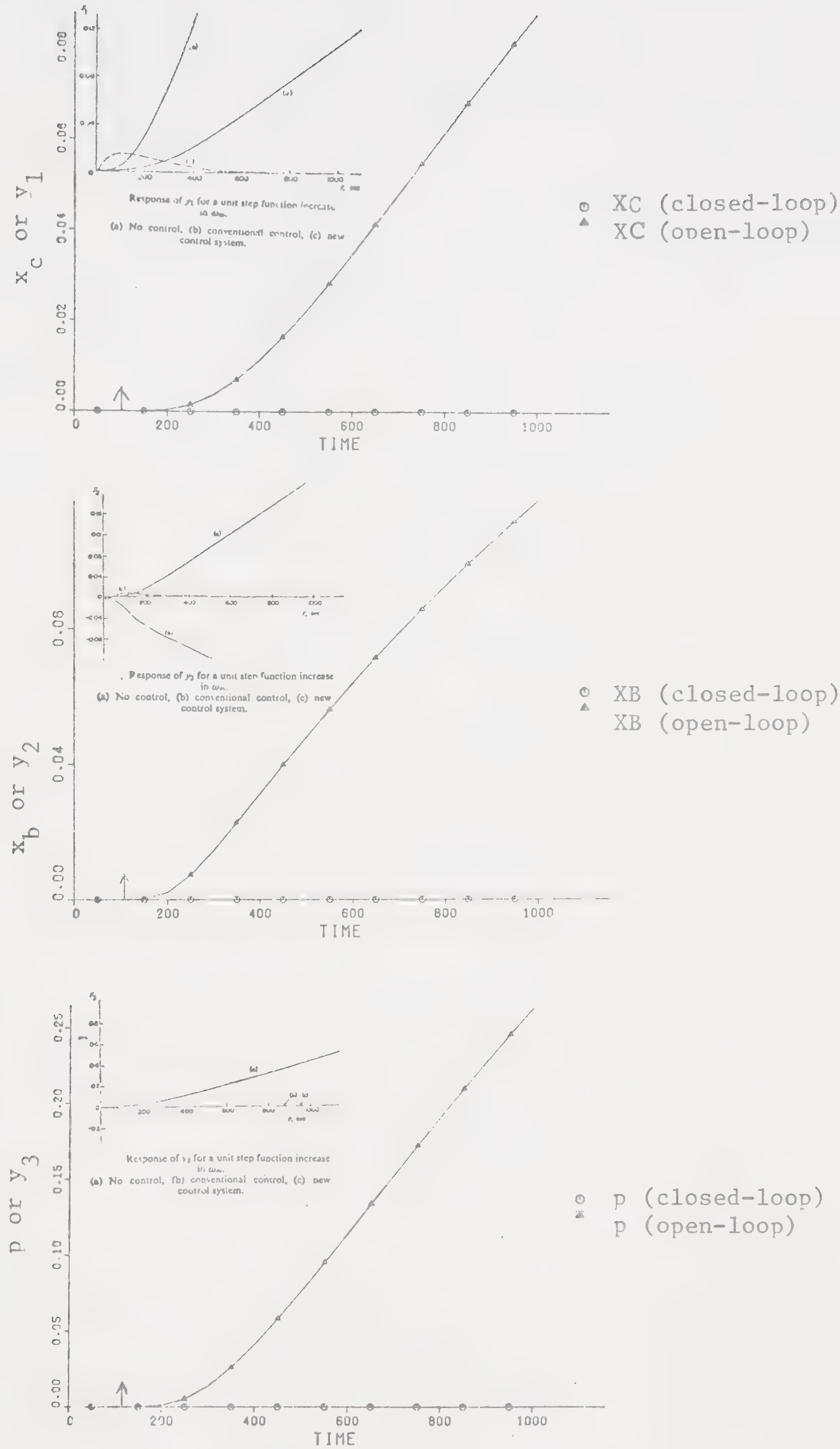


Figure 4.20: Simulated responses of the 11[th] order distillation column model comparing closed-loop and open-loop response of x_c or y_1 (top plot), x_b or y_2 (centre plot), and p or y_3 (bottom plot), to unit step disturbance in in w_m (feed composition).

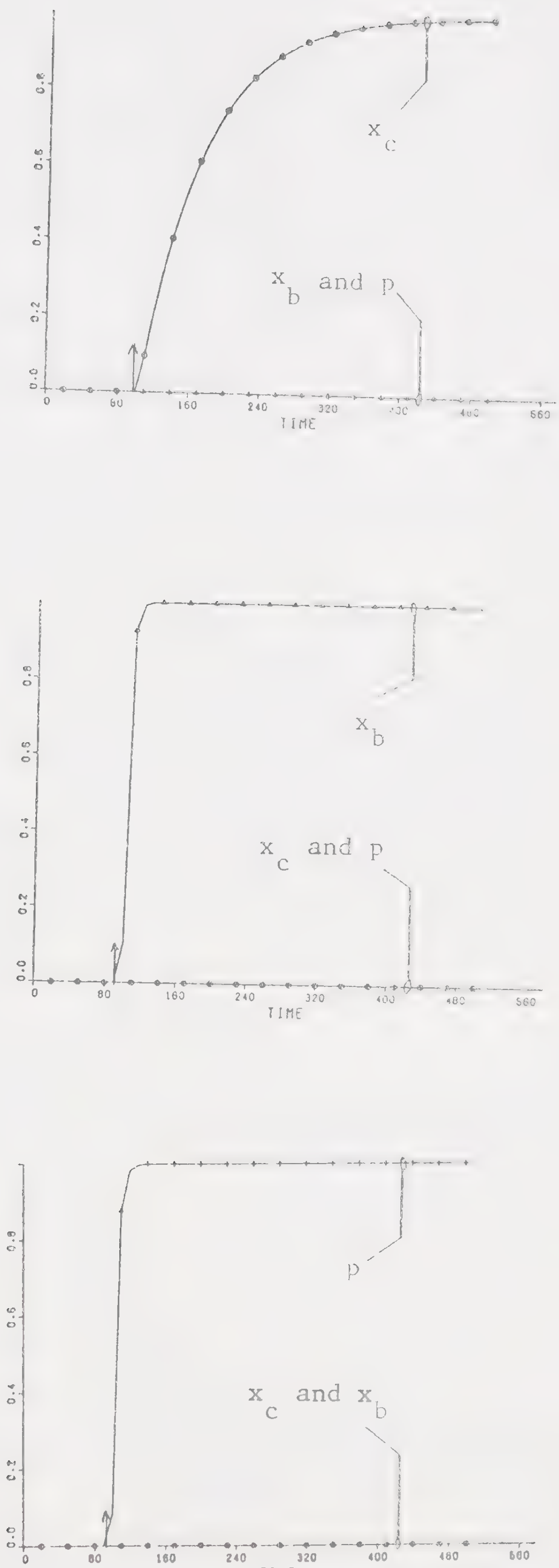


Figure 4.21: Simulated responses of the 11th order distillation column model (with feedback plus setpoint controller), to unit step setpoint changes in x_c or y_1 (top plot), x_b or y_2 (centre plot), and p or y_3 (bottom plot).

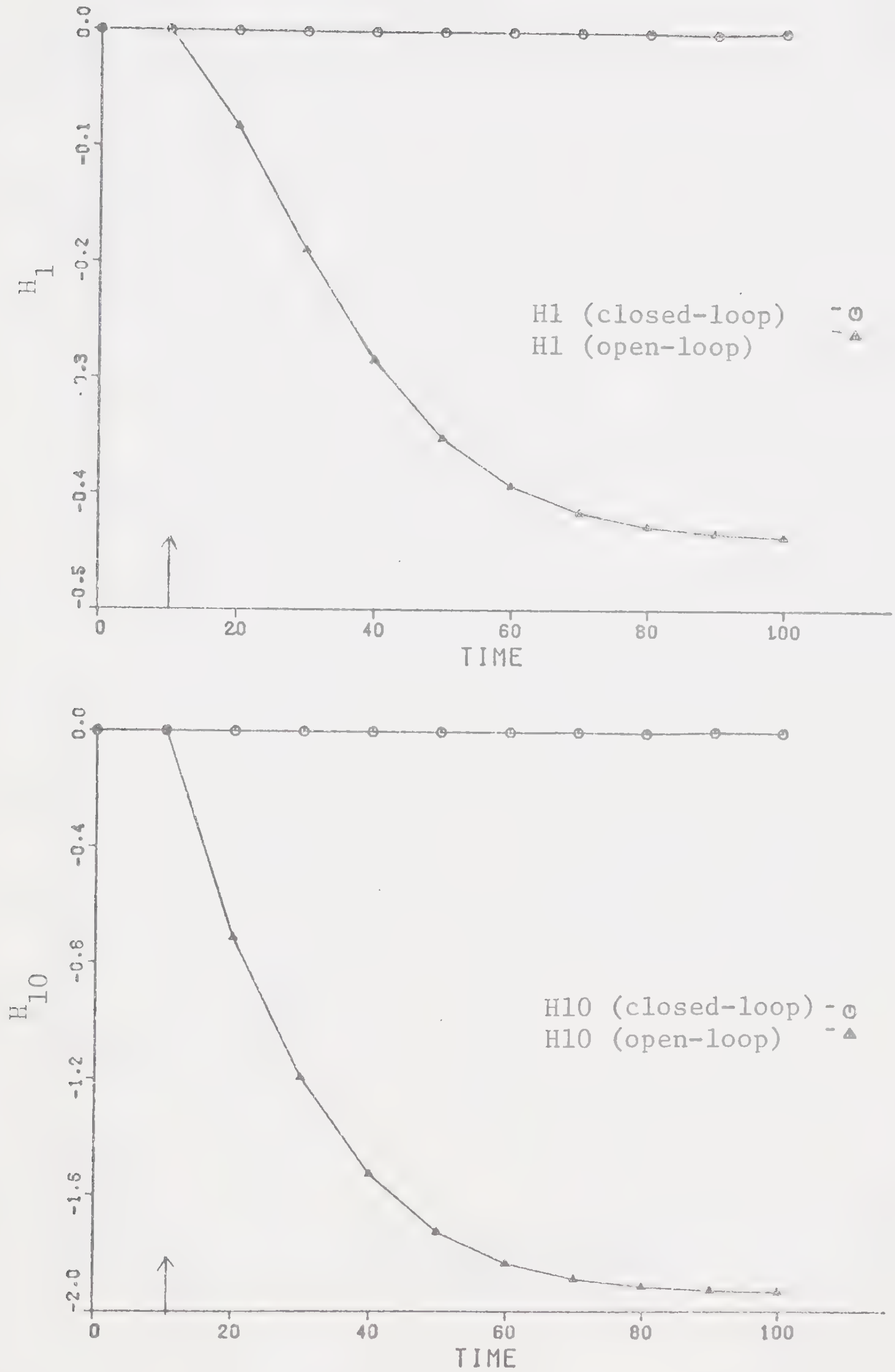
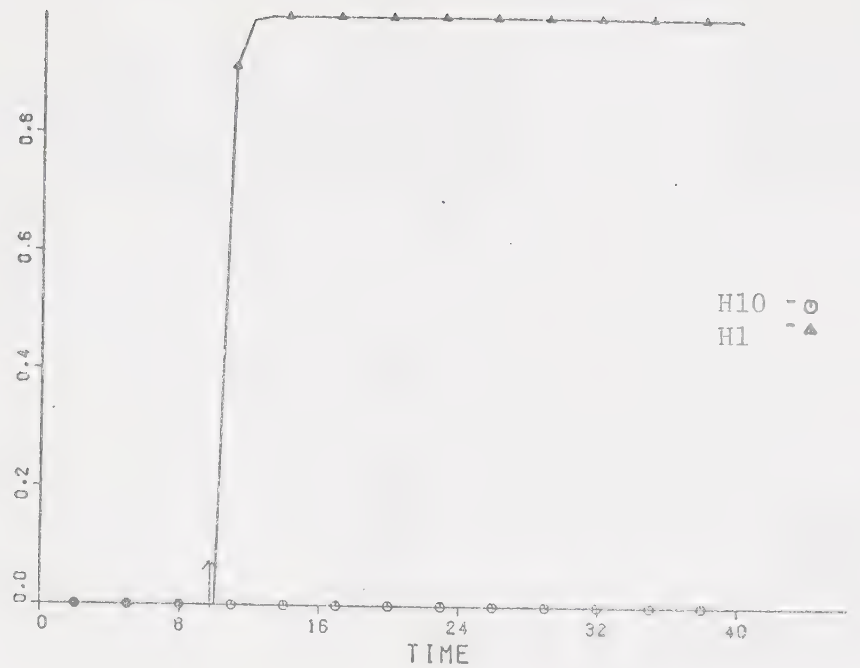


Figure 4.22: Simulated responses of the 20th order distillation column model comparing closed-loop and open-loop response of h_1 (top plot) and h_{10} (bottom plot), to unit step disturbance in feed flowrate.



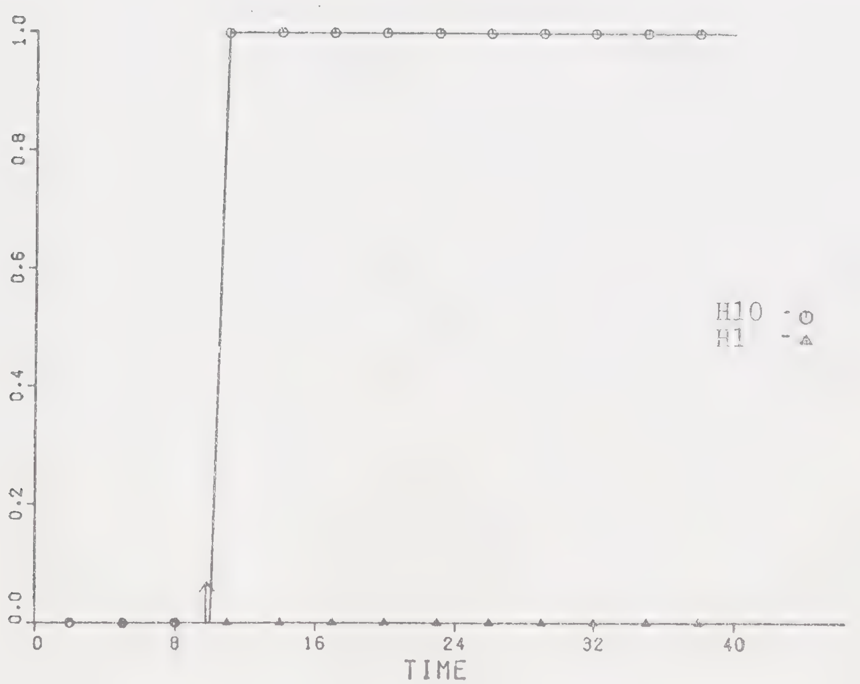


Figure 4.23: Simulated response of the 20^{th} order distillation column model (with feedback plus setpoint controller), to unit step setpoint changes in h_1 (top plot), and h_{10} (bottom plot).

CHAPTER 5

DISTURBANCE MINIMISATION AND POLE ASSIGNMENT IN LINEAR SYSTEMS WITH APPLICATIONS TO OBSERVER DESIGN

Abstract

A method is presented for designing a state feedback controller to assign eigenvalues in a closed-loop system and also minimise a quadratic performance index which includes a measure of coupling between disturbances and the outputs. Two illustrative examples are provided to demonstrate the use of the method. An experimental application of the proposed procedure was carried out on a pilot plant double effect evaporator and this evaluation is also included here. An application of the method to the design of full order observers for systems with unmeasurable disturbances, is also considered.

5.1 Introduction

Practical control systems must operate in environments where unknown external disturbances are invariably present. Despite the inevitability of external disturbances, few multivariable synthesis techniques make use of the information usually available in process models that describes the effect of external disturbances. For example, optimal control systems in which a quadratic performance index is minimised, or conventional pole-placement techniques typically ignore disturbances or only consider impulse-type disturbances. Consequently such systems give satisfactory control when the disturbances occurring in the system are of an impulsive-nature, but may not perform well when the disturbances are sustained, which unfortunately is frequently the case.

The present chapter poses and deals with the following two questions: 1) Is it possible to formulate the regulator problem from an optimisation point of view such that the effect of arbitrary and unknown external disturbances on system outputs of interest is minimised? 2) How does one construct a suitable feedback strategy which provides maximum attenuation of disturbances in the resulting closed-loop system and which ensures that the system eigenvalues are at specified locations in the left half of the complex plane?

The main difference between the approach adopted here and the related results on disturbance localisation (or rejection) by Wonham and Morse [1], Bhattacharyya [2] and in Chapter 3 (see also Shah et al. [3]) is that the results derived here can be applied to any linear, controllable multivariable system, i.e. even systems which do not satisfy the necessary and sufficient conditions for disturbance localisation.

This chapter is organized as follows: Section 5.2 provides an outline of related previous work. Section 5.3 considers the formulation of the performance index or objective function to be minimised in a general form, namely, a functional which is a linear combination of (i) a measure of coupling or transmittance between the disturbances and the outputs, and (ii) a quadratic performance index. Section 5.4 shows how stability and/or arbitrary pole assignment constraints can be incorporated into the above formulation. The design algorithm is also outlined in this section. Section 5.5 contains two numerical examples to demonstrate the use of the proposed design procedure. The application of the design procedure to the design of observers, so that the effect of unmeasurable disturbances is minimised, is the subject of discussion in Section 5.6.

5.2 Previous Work

Wonham and Morse [1], and Bhattacharyya [2] have outlined geometric necessary and sufficient conditions for the existence of state feedback control to eliminate the effect of external disturbances on system outputs. In Chapter 3, the term undisturbable refers to a system in which selected state variables are invariant to arbitrary changes in a specified disturbance, and necessary and sufficient conditions for undisturbability are expressed in terms of the structure of the coefficient matrices in the state space model.

By a geometrical characterization of the regulator problem, Bhattacharyya et al. [4] have derived conditions under which an initially zero output of a system is maintained at zero, and any non-zero output due to non-zero initial conditions or impulse-type disturbances is made to approach zero. An extensive treatment of the regulator problem based on an algebraic approach can also be found in Silverman and Payne [5]. Johnson [6,7] has also studied, from an optimisation point of view, the regulation problem in the presence of a class of unknown disturbances that can be characterized by certain waveform types or modes.

Davison and Smith [8] have derived necessary and

sufficient conditions for the existence of a minimal order, realizable, state feedback controller such that the eigenvalues of the closed-loop system take on pre-assigned values in the left complex plane, and such that the outputs tend to zero as $t \to \infty$, in the presence of constant unknown disturbances. In a later paper, Davison [9] has extended the solution of the above problem to also take into account a class of unmeasurable arbitrary disturbances that satisfy a differential equation. In a more recent paper Willems [10] has derived a set of criteria such that the effect on the output of unmeasurable arbitrary disturbances can be reduced as much as desired by means of suitable stabilizing state feedback.

5.3 Statement of the Problem

Consider the following linear time-invariant multivariable controllable system

$$\begin{aligned} \dot{x} &= Ax + Bu + D\xi \\ y &= Cx \end{aligned} \tag{5.1}$$

where $x \in R^n$ is the state, $u \in R^r$ is the input, $\xi \in R^q$ is the disturbance and $y \in R^m$ is the output; A, B, C and D are constant matrices of appropriate dimensions. For the system in eqn. (5.1) let the control objective be to

design or construct a state feedback matrix K, to generate a control

$$u = Kx \qquad (5.2)$$

so that (i) a specified performance index or an objective functional is minimised and (ii) the resulting closed-loop system is stable. It is well known that of the r x n degrees of freedom available in the design of a state feedback matrix for a controllable system, only n elements, or degrees of freedom are required to assign all n eigenvalues of the system to the desired locations. This leaves (r x n - n) free elements or degrees of freedom in K that can be utilized towards minimising a specified performance index. For the regulatory control problem a widely applicable and useful form of a performance index is a linear combination of a quadratic performance index and an index or measure of coupling between the disturbances and the outputs. An obvious index or measure of coupling between a disturbance, ξ_i, and output, y_j, is the magnitude of the transmittance between ξ_i and y_j. Therefore in its general form the objective functional to be minimised can be represented by:

$$J = \alpha_L J_L + \alpha_q J_q \qquad (5.3)$$

where J_L is a measure of the coupling between the disturbances and outputs, J_q is a quadratic performance criterion and α_L, α_q are weighting factors. J_L is defined as:

$$J_L = \text{trace}\ \left(\sum_{i=1}^{\ell} \beta_i \left|G_L(j\omega_i)^T G_L(j\omega_i)\right|\right) \qquad (5.4)$$

where || denotes absolute value of elements of $G_L{}^T G_L$; the index i=1, ..., ℓ denotes the range of frequencies ω_1, ω_2, ...,ω_ℓ over which the objective functional is to be minimised; β_i is the weighting factor to take into account the contribution of each measure of disturbance/output transmittance or coupling at a particular frequency; and $G_L(j\omega_i)$ is the weighted transmittance at frequency ω_i between the disturbances ξ and outputs y, and in its general form is defined as:

$$G_L(j\omega_i) = P_i C(j\omega_i I - (A + BK))^{-1} DM_i \qquad (5.5)$$

where P_i and M_i are m x m and q x q weighting matrices for the outputs and the disturbances, respectively. By assuming a known dynamic model of the disturbances, Muller and Luckel [22] have also considered the design of state feedback controllers that minimise the effect of external disturbances on the state variables of the system. They have quantified the influence of disturbances by minimising a measure of modal disturbability -

that is the inner product of the reciprocal eigenvectors of the closed-loop system and the disturbance vector, d_i of interest.* The well known quadratic performance index, J_q, is:

$$J_q = \int_0^\infty (x^T Qx + u^T Ru)\,dt \tag{5.6}$$

where Q is a symmetric, positive, semidefinite matrix, and R is a symmetric, positive, definite matrix. A remark regarding the form of the system defined by eqns. (5.1) and (5.2) and the form of J_L and J_q defined by eqns. (5.4) and (5.6) is in order here.

<u>Remark 5.1</u>: The control law as defined by eqn. (5.2) and the corresponding performance index, J, as defined by eqn. (5.3) are in no way limited to a closed-loop system with only proportional state feedback. Dynamic (integral) feedback compensation is easily considered by simply augmenting the original system of eqn. (5.1) to include additional state variables representing, for example, integrals of the outputs.

Note also that in general an objective functional such as the one defined by eqn. (5.4) suffers from the following disadvantage: For finite values of J_L to exist, stability conditions need to be incorporated into the

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* Note: a zero row in Vd_i implies an undisturbable mode.

the functional defined by eqn. (5.3). This is the main subject of discussion in the following section.

5.4 Stability Considerations

To incorporate stability constraints into the system of eqn. (5.1) and (5.2) use is made of an important property of the transformation of a system to a phase-variance canonical form, that is the invariance of the transformation under state feedback. The ensuing analysis briefly follows the approach adopted by Gourishankar and Ramar [11,12] in their minimisation procedure for eigenvalue sensitivity, and also, indirectly, the eigenvalue assignment procedure of Topaloglu and Seborg [13].

By considering state feedback of the form in eqn. (5.2), but one that only utilizes (r - 1) control inputs, i.e. a control law of the form $u' = K'x$ where $u' = [u_1, u_2 \dots u_{j-1}, u_{j+1} \dots u_r]^T$ and $K' = [k_1, k_2 \dots k_{j-1}, k_{j+1} \dots k_r]^T$, the system of eqn. (5.1) can be put into the following single input form:

$$\dot{x} = (A + \sum_{\substack{i=1 \\ i \neq j}}^{r} b_i k_i^T)x + b_j u_j + D\xi = Ex + b_j u_j + D\xi \tag{5.7}$$

where b_i is the i^{th} column of B, and k_i^T is the i^{th} row of K. If now the single input system as characterized by the pair (E,b_j) is controllable and feedback via u_j is admitted, then by a choice of suitable k_j, the resulting system can be assigned desired eigenvalues. It is known that almost any feedback matrix K' will make the system of eqn. (5.7) controllable with respect to input u_j (Davison and Wang [14]). The resulting closed-loop system matrix, H, is then given by:

$$H = (A + \sum_{\substack{i=1 \\ i\neq j}}^{r} b_i k_i^T + b_j k_j^T) = (E + b_j k_j^T) \quad (5.8)$$

Now denote by matrix T(K') (i.e. a matrix T which is a function of K') the nonsingular transformation that transforms the matrix pair (E,b_j) to a phase-variable canonical form. Wilkie and Perkins [15] have shown that such a transformation matrix is invariant under state feedback In the present investigation this fact was found to be useful for eigenvalue assignment in the single input system (E,b_j). Note that there are many other ways of assigning eigenvalues for the pair (E,b_j) (for example via eqn. (2.9)),and then evaluating H. However, in this case the procedure via the phase-variable canonical form was found to be convenient, with H given by the following equation:

$$H = T(K')\tilde{H}G(K')$$

Where for a desired set of closed-loop eigenvalues $\tilde{H}$ is

the phase-variable form of H, and $G(K') = T(K')^{-1}$. With H defined by eqn. (5.9), J_L can be evaluated from:

$$J_L = \mathrm{trace}\left(\sum_{i=1}^{\ell} \beta_i \left| (P_iC((j\omega_i)I-H)^{-1}DM_i)^T (P_iC((j\omega_i)I-H)^{-1}DM_i) \right|\right) \tag{5.10}$$

Likewise J_q can also be evaluated as a function of K', Ramar and Gourishankar [12]. With the control law of the form:

$$u' = K'x \tag{5.11}$$

the quadratic performance index, $\hat{J}_q(K')$ is given by:

$$\hat{J}_q(K') = \int_0^{\infty} (x^TQx + u'^TRu')dt = \int_0^{\infty} x^T(Q + K'^TRK')xdt \tag{5.12}$$

For initial conditions $x(0) = x_o$, $\hat{J}_q(K')$ can be expressed as

$$\hat{J}_q(K') = x_o^T V(K')x_o \tag{5.13}$$

where V(K') is the symmetric positive-definite solution of the matrix Lyapunov equation

$$V(K')H + H^TV(K') + Q + K'^TRK' = 0 \tag{5.14}$$

A criterion which is not dependent on the initial conditions x_o and is 'optimal' in the average sense can be

considered by modifying (5.13) to [25]:

$$J_q(K') = \text{trace}[V(K')] \qquad (5.15)$$

Thus $J_q(K')$ can be evaluated for different values of K' by solving for $V(K')$ in eqn. (5.14).

5.4.1 The design algorithm

Thus, very simply, the main step in the design procedure for minimising J requires J_L and J_q to be computed for a given set of desired eigenvalues and a choice of gain parameters in K' as determined by a numerical search procedure. In detail the algorithm consists of the following steps:

i) An initial choice of K' is made.

ii) $T(K')$ and $G(K')$ are evaluated for the matrix pair (E, b_j).

iii) With the desired closed-loop eigenvalues specified, the closed-loop system characteristic polynomial and hence $\tilde{H}$ are evaluated. H is then computed from eqn. (5.9).

iv) J_L is computed from eqn. (5.10) and J_q is computed from eqn. (5.14) and (5.15).

v) A numerical search procedure is used to estimate a new set of values for K' and steps (ii) to (iv) repeated until a K'_{opt} is found that gives a

minimum value of $J = J_L + J_q$.

vi) k_j is computed from the final $T(K'_{opt})$ and the desired set of closed-loop eigenvalues to yield a final K_{opt}.

Note that unless $J = 0$ the final K_{opt} that is selected is not truly optimal, since J is also a function of b_j and the measure of controllability (eg. cf. Simon and Mitter [26]). Thus for the true optimal solution the procedure is repeated for $j = 1, \ldots r$ (assuming that (E, b_j) is always controllable) and the final K_{opt} is selected to be that corresponding to the minimum of $J(b_j)$, $j = 1, \ldots r$.

It is easy to extend the minimisation procedure proposed above to the design of a unity rank feedback matrix of the type: $K = gf^T$, where g is an r x 1 vector and f is an n x 1 vector. In this case the minimisation procedure selects parameters in the vector g to minimise the performance index defined in eqn. (5.3), while the vector f is used to assign desired eigenvalues. Such a procedure has clear advantages over conventional pole-placement schemes where the choice of g depends only on the controllability of the pair (A, Bg) and is otherwise arbitrary. In addition this design procedure can be used as an alternative when the minimisation procedure for the design of nonunity rank feedback matrix

cannot be easily applied to high order systems because of computational difficulties usually encountered in high dimensional search problems.

5.4.2 Disturbance localisation vs. disturbance minimisation

The time response of the i^{th} system output, $y_i(t)$, in the presence of external disturbance, ξ_j, with the initial state vector, $x(0) = 0$, is given by:

$$y_i(t) = \int_o^t \sum_{\ell=1}^{n} c_i^T w_\ell v_\ell^T d_j \exp\{\lambda_\ell (t - \tau)\}\xi_j(\tau)d\tau \tag{5.16}$$

where c_i^T is the i^{th} row of C, w_ℓ and v_ℓ^T are the ℓ^{th} eigenvector and reciprocal eigenvector of H, d_j is the j^{th} column of D, λ_ℓ is the ℓ^{th} eigenvalue and ξ_j is the j^{th} component of disturbance, ξ. The necessary and sufficient condition for disturbance localisation or invariance of $y_i(t)$ in the presence of $\xi_j(t)$ is, Shah et al. [16] (cf. Chapter 3).

$$c_i^T w_\ell v_\ell^T d_j = 0 \quad \text{for all} \quad \ell = 1, 2, \ldots n \tag{5.17}$$

In contrast to this, disturbance minimisation over a wide range of frequencies requires that the expression of eqn.

(5.4) be a minimum. For example, $J_L = 0$ at $s = j\omega = 0$ implies asymptotic tracking, in the presence of step disturbances this is

$$\sum_{\ell=1}^{n} c_i^T w_\ell v_\ell^T \left(-\frac{1}{\lambda_\ell}\right) d_j = 0 \tag{5.18}$$

Note that frequently for low order systems with $r = n$ it is possible to carry out the minimisation scheme analytically for $s = 0$ by using expression (5.18) and thus achieve zero offset without integral control.

5.5 Illustrative Examples

To demonstrate the use of the proposed design procedure, a numerical search procedure of Fletcher [17] as based on a quasi- Newton method (and available as subroutine ZXMIN in the International Mathematical and Statistical Libraries, IMSL Library 1, Edition 5, 1975), was applied to a second order system with coefficient matrices A, B, D and C given by:

$$A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$$

$$C = I_2 \tag{5.19}$$

To preserve clarity and for the sake of computational

simplicity, consider only the minimisation of J_L; that is, in eqn. (5.3) choose $\alpha_L = 1$ and $\alpha_q = 0$. Furthermore, assume that ξ_1 is the main disturbance variable of interest in the above system so that the main objective is to regulate $y_1 = x_1$ well in the presence of disturbance ξ_1 and to a lesser extent ξ_2. For the purpose of comparison of the present design scheme with other control schemes it was decided to limit the absolute value of the gain elements in the controller to 15.

5.5.1 Proportional control

The values of parameters ℓ, W_ℓ, β_i, P_i, M_i as defined in eqn. (5.4) and the resulting disturbance minimisation controller, K_1, are shown in Table 5.1. Also shown in Table 5.1 is an optimal controller, K_2, designed for the system of eqn. (5.19) by minimising a quadratic performance index. The choice of state and control weighting matrices Q and R respectively is shown in Table 5.1. Since B^{-1} exists for the second order system, it was possible to also design a controller, K_3 by a combination of pole-placement and analytical minimisation techniques. Details of controller, K_3, are also shown in Table 5.1.

Table 5.1

Details of Control Schemes for Proportional State Feedback of the System of Equation (5.19)

Controller-Type	Closed-Loop Eigenvalues	Other Design Details
Disturbance Minimisation $K_1 = \begin{bmatrix} -13.92 & 1.99 \\ 7.56 & 0.92 \end{bmatrix}$	-6.0, -7.0 as specified	For Equation 5.4 $\ell=2$, $\omega_1=0$, $\omega_2=1.0$ rad. t^{-1} $\beta_1=5.0$, $\beta_2=0.1$, $M_1=M_2=I_2$ $P_1=P_2=$ Diag [10 1] Initial choice of $K' = [0 \quad 0]$
Optimal Control $K_2 = \begin{bmatrix} -14.76 & 3.32 \\ 3.32 & -1.99 \end{bmatrix}$	-5.58, -11.17 not specified	Q = Diag [100 1] R = Diag [0.65 0.65]
Pole Placement $K_3 = \begin{bmatrix} -15.0 & 4.0 \\ -2.0 & -6.0 \end{bmatrix}$	-9.0, -12.0 as specified	Because B^{-1} exists, strictly speaking this design scheme consists of a combination of pole placement and an approximate analytical minimisation (cf. Section 5.4.2 and eqn. (5.18)).

Figures 5.1 and 5.2 show the response of $x_1(t)$ to a unit step disturbance in $\xi_2(t)$, respectively, when each of the three feedback matrices K_1, K_2 and K_3 are implemented. In each case, as would be expected, the response of the system with the disturbance minimisation controller K_1 is better or at least comparable to that of the optimal system. Figure 5.3 and 5.4 show the response of $x_2(t)$ to a unit step disturbance in $\xi_1(t)$ and $\xi_2(t)$ respectively. In Figures 5.3 the response of $x_2(t)$ to a unit step disturbance in $\xi_1(t)$ is worse than that of the system with optimal or pole-placement controller. This is intuitively expected since in trying to reduce the effect of $\xi_1(t)$ on $_1(t)$ as required for the design objective with limited gains and therefore limited control energy, the effect of the disturbance is necessarily diverted to the remaining state variable of the system, which in this case is $x_2(t)$.

5.5.2 Proportional plus integral control

By augmenting the original system of eqn. (5.19) to a third order system where $\dot{x}_3 = x_1$, integral feedback of y_1 or x_1 can also be considered. A disturbance minimisation controller, K_4, was designed for the above system and is shown in Table 5.2, together with the

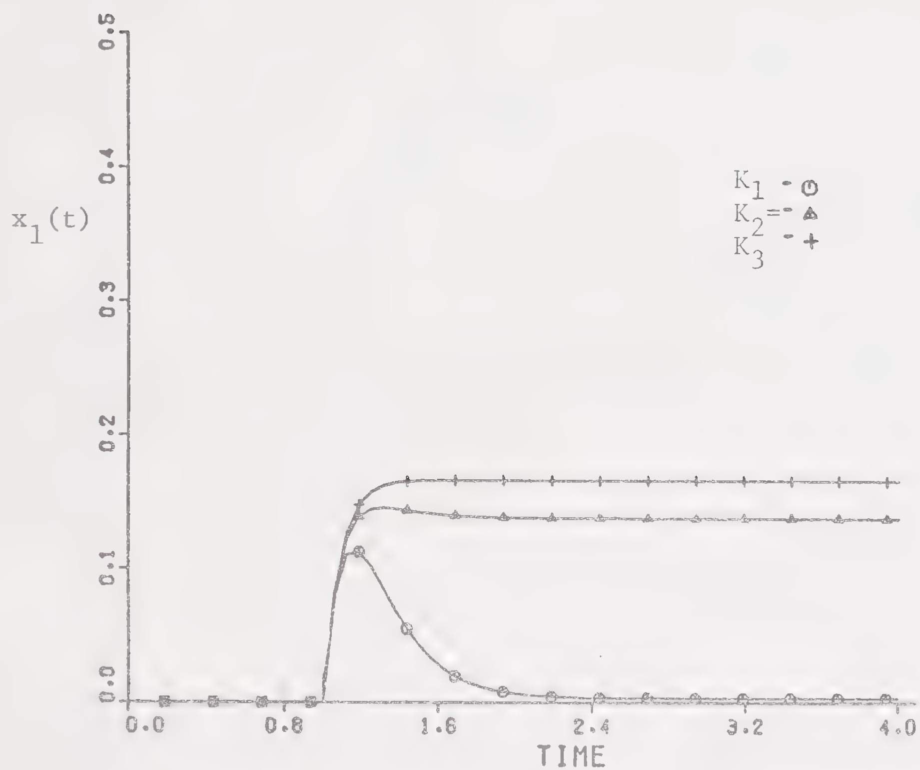

Figure 5.1: Response of $y_1(t)$ to a unit step disturbance in $\xi_1(t)$ at time t=1.0, for the system of eqn. (5.19) with controllers K_1, K_2 and K_3 as shown in Table 5.1.

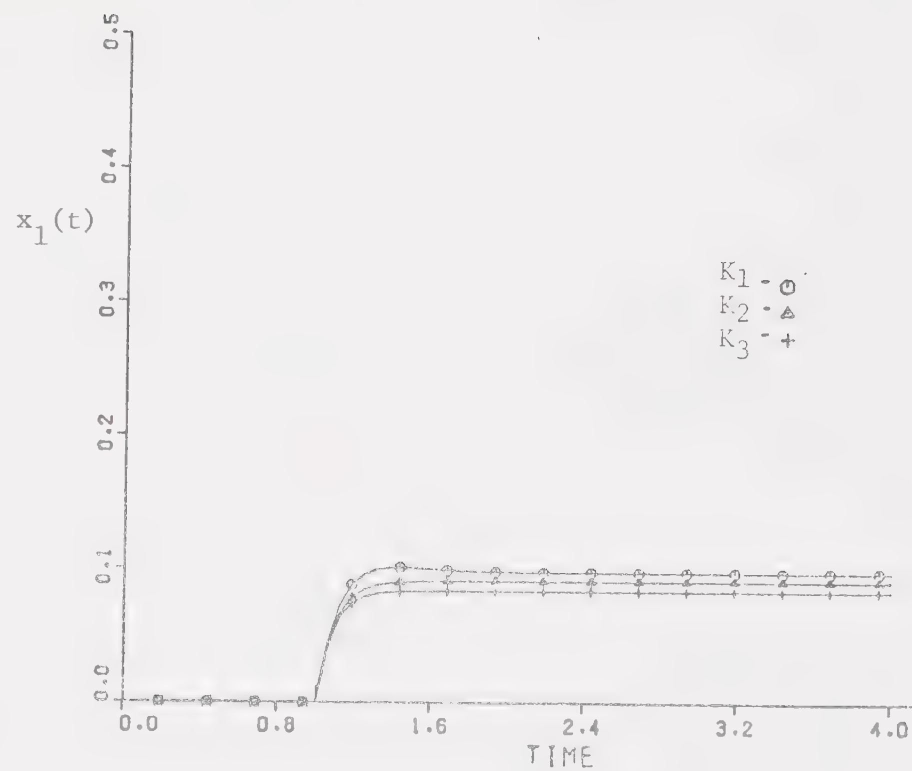

Figure 5.2: Response of $y_1(t)$ to a unit step disturbance in $\xi_2(t)$ at time t=1.0, for the system of eqn. (5.19) with controllers K_1, K_2 and K_3 as shown in Table 5.1.

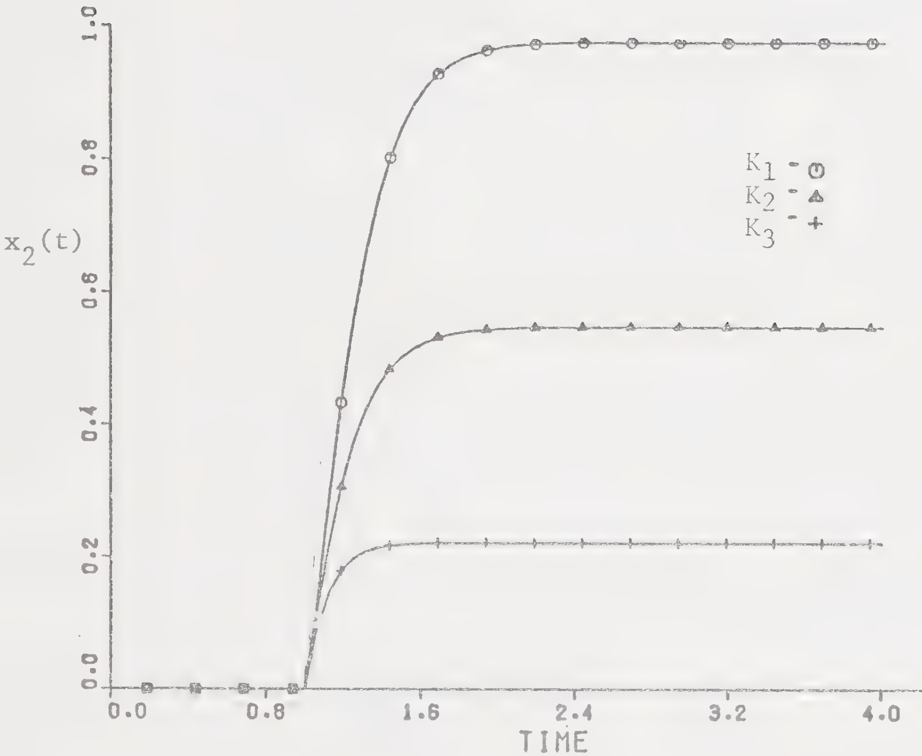

Figure 5.3: Response of $x_2(t)$ to a unit step disturbance in $\xi_1(t)$ at time t=1.0, for the system of eqn. (5.19) with controllers K_1, K_2 and K_3 as shown in Table 5.1.

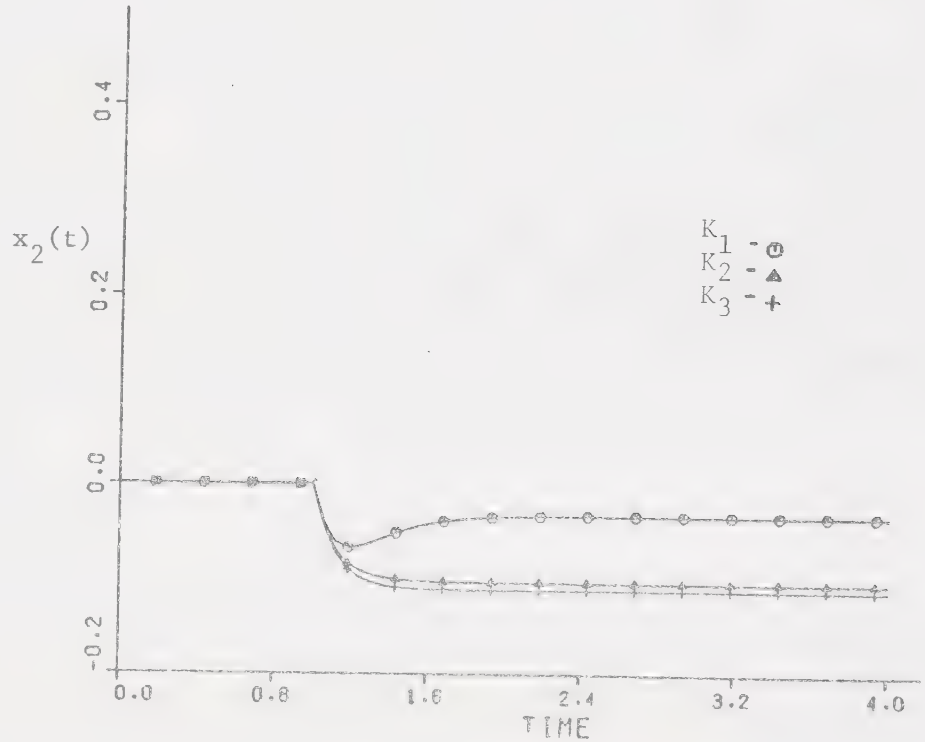

Figure 5.4: Response of $x_2(t)$ to a unit step disturbance in $\xi_2(t)$ at time t=1.0 for the system of eqn. (5.19) with controllers K_1, K_2 and K_3 as shown in Table 5.1.

Table 5.2

Details of Control Schemes for Proportional plus Integral State Feedback for the System of Equation (5.19)

Controller-Type	Closed-Loop Eigenvalues	Other Design Details
Disturbance Minimisation $K_4 = \begin{bmatrix} -14.0 & 1.33 & -14.27 \\ 0.90 & 0 & -0.3 \end{bmatrix}$	-1.0, -6.0, -7.0 as specified	$\ell=2$, $\omega_1=0$, $\omega_2=3.0$ rad. t^{-1}, $\beta_1=2.0$ $\beta_2=0.0001$, $P_1=P_2=\mathrm{diag}[1 \quad 0]$; $C=I_2$ $M_1=I_2, M_2=\mathrm{Diag}\,[10 \quad 5]$. Initial choice non-zero K'. This is probably a local minimum. Difficult to obtain a global minimum.
Optimal Control $K_5 = \begin{bmatrix} -14.84 & 3.38 & -11.45 \\ 3.38 & -1.98 & 1.43 \end{bmatrix}$	-1.0, -5.76, -10.06 not specified	$Q = [1000 \quad 1 \quad 800]$ $R = [7.5 \quad 7.5]$
Conventional Pole-Placement Method $K_6 = \begin{bmatrix} 0 & 0 & 0 \\ 24.5 & -14.0 & 10.5 \end{bmatrix}$	-1.0, -6.0, -7.0 as specified	$K_6 = g\bar{f}^T$ with $g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

continued ...

Table 5.2 (continued)

Controller-Type	Closed-Loop Eigenvalues	Other Design Details
Pole Placement Using Unity Rank Feedback Plus Minimisation $K_7 = \begin{bmatrix} -14 & 0 & -14 \\ 0 & 0 & 0 \end{bmatrix}$	-1.0, -6.0, -7.0 as specified	$K_7 = gf^T$ g chosen by minimisation scheme as $g \approx \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

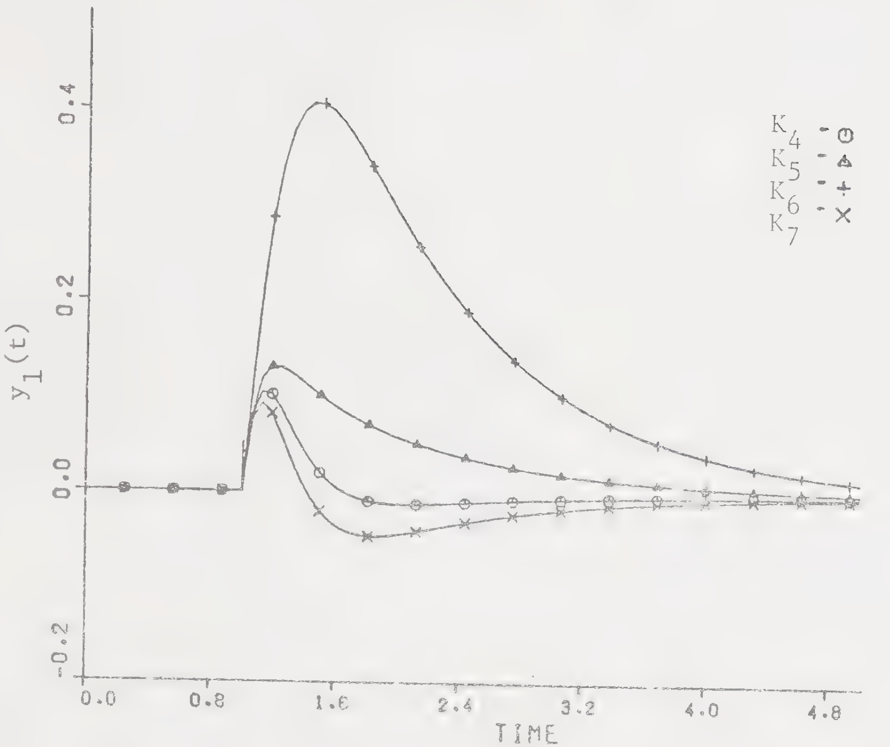

Figure 5.5: Response of $y_1(t)$ to a unit step disturbance in $\xi_1(t)$ at time t=1.0, for PI control with controllers K_4, K_5, K_6 and K_7 as shown in Table 5.2.

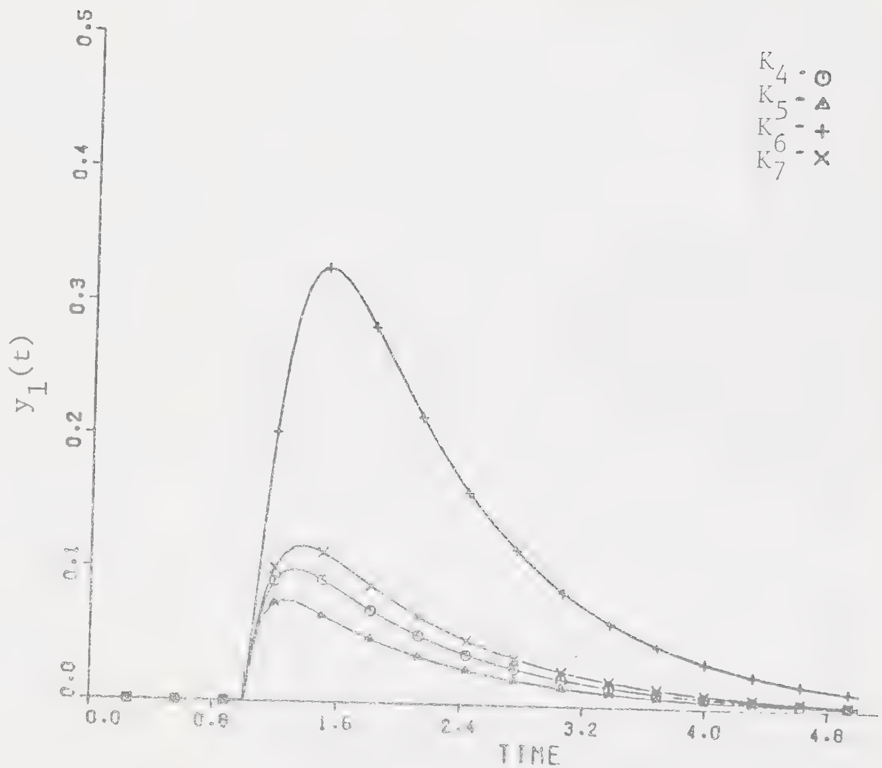

Figure 5.6: Response of $y_1(t)$ to a unit step disturbance in $\xi_2(t)$ at time t=1.0, for PI control with controllers K_4, K_5, K_6 and K_7 as shown in Table 5.2.

values of required parameters as defined in eqn. (5.4). For the purpose of comparison an optimal PI controller, K_5, designed by minimising a quadratic performance index was also calculated and is shown in Table 5.2. Finally two different unity rank state feedback matrices K_6 and K_7 were also designed. These are also shown in Table 5.2. Controller K_6 was designed by using a conventional pole placement scheme with an arbitrary choice of $g = [0\ 1]^T$, where $K_6 = gf^T$. K_7 was designed by application of the minimisation procedure in the optimal choice of the g.

Figure 5.5 and 5.6 show the response of $y_1(t)$ to unit step disturbances in $\xi_1(t)$ and $\xi_2(t)$ respectively. In each case, as would be expected, the disturbance minimisation controller, K_4, outperforms the conventional pole placement controller K_6 and is better than or at least comparable to the performance of the optimal system.

5.6 Applications to Observer Design

An immediate use of the above method is in its application to the dual or adjoint of system of eqn. (5.1) and thus to the design of observers in which the effect of ummeasurable disturbances on the state estimates of interests can be minimised. The method is of

course only directly applicable to the design of identity, or full-order observers, and not to the design of reduced order observers which have necessarily limited degrees of freedom. The flexibility offered by the method in allowing arbitrary eigenvalue placement is especially useful here.

Johnson has considered the synthesis problem of unknown input observers by modelling the unknown inputs by a dynamical system or approximating them by a polynomial function; all of his results have been succintly summarized in [18].

Basile and Marro [19] have discussed the design of full order observers for systems with unknown inputs and disturbances. Their method requires a recursive algorithm for design, but the procedure for determination of stability conditions requires a complex algorithm. Mita [20] has recently proposed a method for the design of a reduced order observers for multivariable systems such that the required state estimates are made insensitive to unknown inputs or disturbances.

More recently Gourishankar et al. [21] have also considered the problem of designing observers for systems subjected to unmeasurable inputs that can be approximated by a polynomial type function. They have also derived a set of necessary and sufficient

conditions that can be readily applied to determine the existence of an observer for such a case.

For completeness the main steps involved in the design of an n^{th} order observer for the system of eqn. (5.1) are now summarized:

i) If the output matrix C in the system of eqn. (5.1) does not satisfy $C = [I_m O]$ then the system of eqn. (5.1) is transformed such that the final transformed system has an output matrix C', where $C' = [I_m O]$. For convenience of notation drop the superscript ' and always assume from here on that C is in the required form.

ii) Let the dynamics of the observer be governed by the following equation:

$$\dot{z} = Fz + Gy + Bu \tag{5.20}$$

with $F = A - GC$, and the required estimate, y_z, defined by $y_z \triangleq [\hat{x}_{m+1}, \hat{x}_{m+2}, \ldots, \hat{x}_n]^T \triangleq [z_{m+1}, z_{m+2}, \ldots, z_n]^T$. (Note that $z^T = [y \; y_z]$.) Thus the observer output equation is

$$y_z = Nz, \text{ where } N^T \text{ is a basis matrix for Ker C.} \tag{5.21}$$

iii) The error dynamics of actual states $x_{m+1}, x_{m+2}, \ldots, x_n$ and the observer states $z_{m+1}, z_{m+2}, \ldots, z_n$

are given by the transfer function matrix:

$$e(s) = N\ [sI - (A - GC)]^{-1} D\ \xi(s) \qquad (5.22)$$

where $e(t) = [(x_{m+1}(t) - z_{m+1}(t), (x_{m+2}(t) - z_{m+2}(t)), \ldots, (x_n(t) - z_n(t))]$. Thus for a system with state matrix A^T, input matrix C^T, disturbance matrix N^T, and output matrix D^T, the method of Sections 5.3 and 5.4 can be applied directly to construct G^T such that the following measure of coupling or transmittance between $\xi(s)$ and $e(s)$ is minimised:

$$J_{LO} = \text{trace}\ (\sum_{i=1}^{\ell} \beta_i |(P_i D^T (sI - (A^T - C^T G^T))^{-1} N^T M_i)^T$$

$$P_i D^T (sI - (AT - C^T G^T))^{-1} N^T M_i)| \qquad (5.23)$$

5.6.1 Example

Consider the following state space matrix A, the output matrix C and the disturbance matrix D of a third order system for which an observer to estimate $x_1(t)$ is to be designed.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{and } D = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \tag{5.24}$$

If the disturbance ξ_1 is unmeasurable, then it is clear from the error dynamics of the full order observer (cf. eqn. (5.22)) that the error in the estimate of $x_1(t)$ has a non-zero value for constant disturbances. Thus, let the main objective in the design of an observer for the system of eqn. (5.24) be to obtain a good estimate of $x_1(t)$ even in the presence of unknown disturbances in $\xi_1(t)$.

By considering the dual of the above system where the state matrix is A^T, input matrix is C^T, output matrix is D^T and disturbance matrix is $[1\ 0\ 0]^T$, the proposed procedure for disturbance minimisation can be applied to determine an optimum observer gain matrix, G. Three different observer gain matrices, G_1, G_2 and G_3 were designed incorporating different design parameters and specified closed-loop eigenvalues. Values of G_1, G_2 and G_3 and the corresponding details of the

parameter values used in the design scheme are shown in Table 5.3. Figure 5.7 shows the error $e_1(t)$ for the estimate of $x_1(t)$, in the presence of a unit step disturbance in $\xi_1(t)$. The time response of $e_1(t)$ in Figure 5.7 graphically illustrates the design freedom available in the disturbance minimisation scheme, namely, arbitrary eigenvalue assignment plus minimisation of disturbances at one or more frequencies. To compare the design of the observer gain matrix by a disturbance minimisation scheme to other conventional design schemes for a full order observer, the optimal control scheme was applied to the dual of the system of eqn. (5.24). The resulting gain matrix G_4 as well as the weighting matrices Q and R are shown in Table 5.3. A reduced order observer was also designed and the resulting gain matrix G_5 with other design details is shown in Table 5.3. Figure 5.8 shows the error $e_1(t)$ in the presence of a unit step disturbance in $\xi_1(t)$ when three different gain matrices G_3, G_4 and G_5 are implemented for the system of eqn. (5.24). As would be expected, the gain matrix G_3, designed by using the disturbance minimisation scheme performs better than the two other schemes.

Table 5.3

Details of the Observer Gain Matrices for the System of Equation (5.24)

Gain Matrix-Type	Eigenvalues	Design Details
Disturbance Minimisation $G_1 = -\begin{bmatrix} -4.62 & 4.14 \\ -3.30 & 0.53 \\ -3.92 & 4.34 \end{bmatrix}$	-1.0, -2.0, -3.0 as specified	$\ell = 1$, $\omega_1 = 0$, $\beta_1 = 5.0$ Initial choice of K' zero. Probably only a local minimum.
Disturbance Minimisation $G_2 = -\begin{bmatrix} -9.78 & 4.41 \\ -6.06 & 0.19 \\ -2.42 & 4.25 \end{bmatrix}$	-2.0, -3.0, -5.0 as specified	$\ell = 1$, $\omega_1 = 0$, $\beta_1 = 5.0$ Initial choice of K' non-zero. Probably only a local minimum.
Disturbance Minimisation $G_3 = -\begin{bmatrix} -2.67 & 2.39 \\ -0.85 & -3.00 \\ -4.34 & 8.50 \end{bmatrix}$	-1.0, -2.0, -5.0 as specified	$\ell = 2$, $\omega_1 = 0$, $\omega_2 = 0.2$, $\beta_1 = 5.0$, $\beta_2 = 1.0$. Initial choice of K' is zero. Probably only a local minimum.

Table 5.3 (continued)

Gain Matrix-Type	Eigenvalues	Design Details
Optimal Control $G_4 = -\begin{bmatrix} -2.56 & -6.58 \\ -0.05 & -4.82 \\ -10.53 & 6.53 \end{bmatrix}$	$-1.80 \pm j1.63$, -12.31 not specified	$Q = [100 \quad 1 \quad 200]$ $R = [2.0 \quad 2.0]$
Reduced Order Observer $G_5 = [-5.0 \quad 5.0]$	-10.0 as specified	Error dynamics of the reduced order observer are given by: $\dot{e}_1(t) = (C'AL_2 - G_5CAL_2)e(t)$ $+ C'D\xi(t)$ where $C' = [1 \quad 0 \quad 0]$, $L_2 = [1 \quad 0 \quad -1]^T$

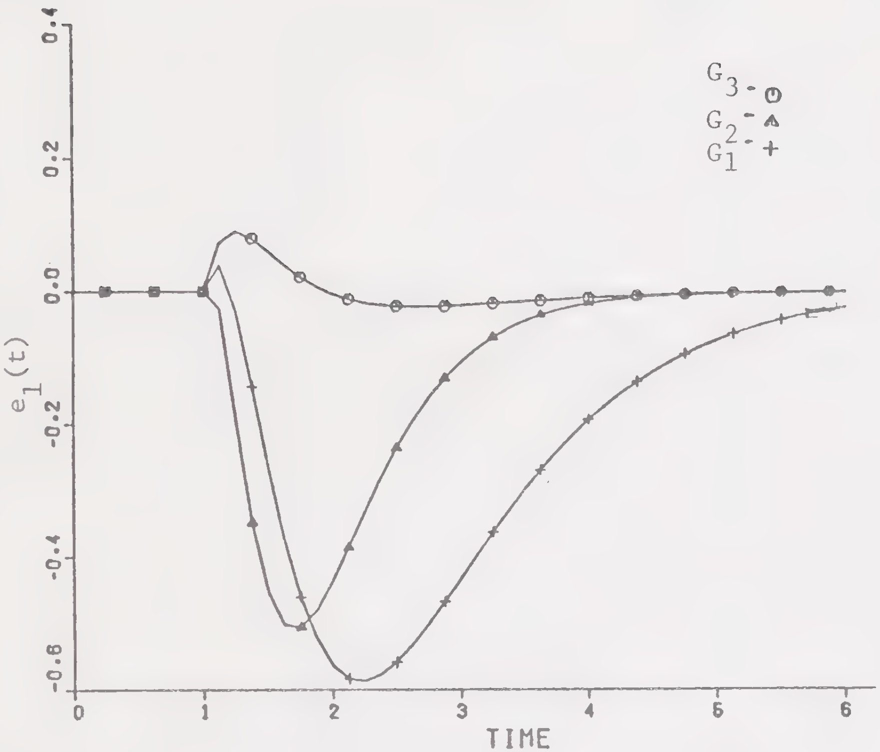

Figure 5.7: The error response $e_1(t)$ in the estimate of $x_1(t)$ using a full order observer, when the system of eqn. (5.24) is subjected to a unit step disturbance in $\xi_1(t)$ at time t=1.0. Controllers G_1, G_2, G_3 are as shown in Table 5.3.

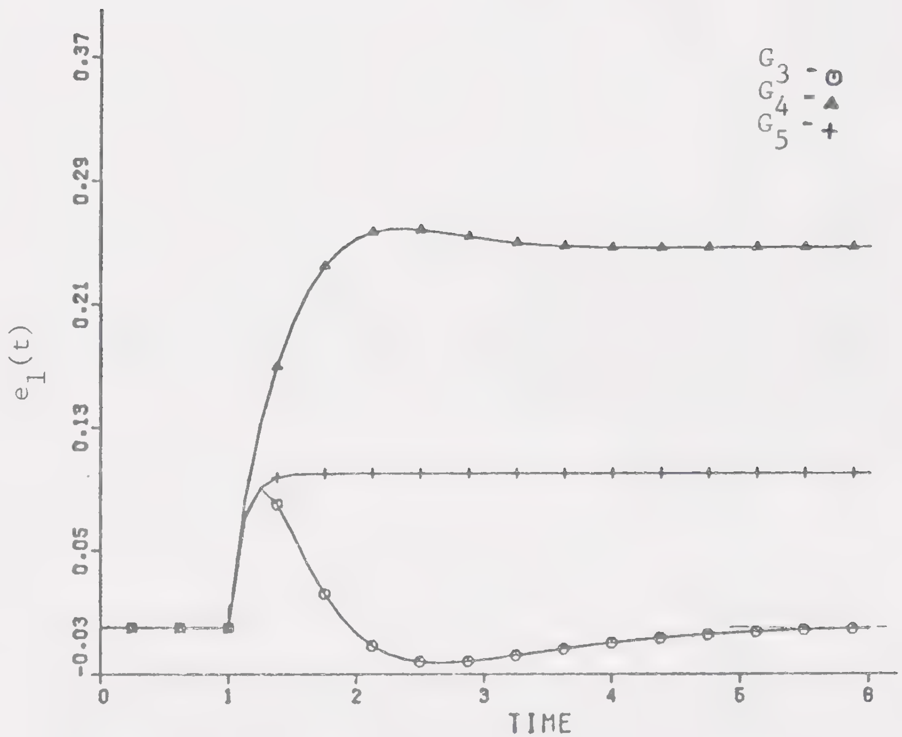

Figure 5.8: The error response $e_1(t)$ in the estimate of $x_1(t)$ using a full order observer, when the system of eqn. (5.24) is subjected to a unit step disturbance in $\xi_1(t)$ at time t=1.0. Controllers G_3 G_4, G_5 are as shown in Table 5.3.

5.7 Experimental Evaluation

The design approach developed here to minimise the effect of external disturbances on outputs of interest was evaluated experimentally by designing a controller for the reduced third-order model of a computer-controlled, pilot-plant evaporator at the University of Alberta. A schematic diagram of the evaporator, with details on its mode of operation, and a table of important variables together with its discrete third-order state space model, are given in Appendix C. This evaporator has been used previously to evaluate a number of modern multivariable control methods (Fisher and Seborg,[23]) and hence it is possible to make direct comparisons between alternative control techniques. The third order evaporator model has 3 state variables (which are also the 3 output variables), 3 inputs and 3 disturbances.

The main output variable of interest is the second-effect or product concentration, C2. Previous operating experience has shown that the most frequent and severe disturbances in product concentration, C2, are produced by variations in feed flowrate, F. Therefore for the purpose of this application the design objectives for the controller are:

a) to minimise the effect of feed disturbance, F, on the product concentration, C2, and

b) to assign all closed-loop eigenvalues closer to the origin.

A simple inspection of matrices Φ,Δ (for the third order model as tabulated in Appendix C) reveals that the input variable, B2, can only control or manipulate the state variable, W2, i.e. C2 and W1 are structurally uncontrollable by B2. For this reason B2 was first used to assign the eigenvalue of the mode corresponding to W2. The resulting 3rd order system, with 2 inputs, 3 disturbances and B2 = 13.05*W2 is shown below:

$$\Phi' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.47 & 0 \\ 0 & 0 & 0.96 \end{bmatrix}, \qquad \Delta' = \begin{bmatrix} -0.0325 & -0.0811 \\ -0.0377 & 0.0854 \\ 0.0527 & -0.0441 \end{bmatrix}$$

$$\theta = \begin{bmatrix} 0.120 & 0 & -0.0135 \\ 0.0032 & 0 & -0.0156 \\ -0.0218 & 0.0398 & 0.0218 \end{bmatrix}$$

The above step serves two main purposes. First, it changes the analytically simple control problem (because Δ can be inverted) into a not so straightforward problem. This is useful in demonstrating the practicality

of the method. Secondly, it reduces the dimensionality of the numerical search procedure from 6 to 3, which is computationally easier to handle. With the two control objectives; a) and b) in mind, the proposed design procedure was used to minimise the effect of disturbance , F, on C2, at steady state, i.e. at $\omega = 0$. The desired eigenvalues of the closed-loop system were chosen as 0.65, 0.47 and 0.28. The resulting controller, K_{dmin}, is as shown in Table 5.4. Also shown in Table 5.4 is controller, K_{opt}, designed by using optimal control method, Wilson [24]. The values chosen for matrices, Q and R in computing K_{opt} are also shown in Table 5.4.

The response of the closed-loop system to a ± 20% disturbance in feed flow rate with controller 5 K_{dmin} and K_{opt} implemented is shown in Figures 5.9 and 5.10 respectively. As expected, the controller K_{dmin} reduces the steady state offset in C2 to almost zero. The performance of this controller is better than or at least as good as the performance of the optimal controller K_{opt}. Operating experience on the evaporator has shown that disturbances in feed concentation, CF, does not significantly affect the product concentration, C2. To check if minimising the effect of F on C2 had not made C2 more sensitive to disturbances in CF, a

TABLE 5.4

Details of a disturbance minimisation controller, K_{dmin}, and an optimal controller, K_{opt}, based on the third order discrete state space model of the evaporator.

Run No.*	Controller	Eigenvalues and other design details	Experimental run conditions
DMIN/P/1	$K_{dmin} = \begin{bmatrix} 3.70 & 0.06 & -9.0 \\ 3.02 & -0.21 & 3.88 \\ 0 & 13.05 & 0 \end{bmatrix}$	Λ = diag[.28,.47,.65] Minimisation of effect of F on C2 at ω = 0 P=M=I; C=[0 0 1] K'(0)=[2.0 0 0 -9.0] from $K_{d\ell 1}$ (Table 4.1)	±20% disturbance in feed flowrate (Figure 5.9); ∓30% disturbance in feed concentration (Figure 5.11).
OPT1/P/1	$K_{opt1} = \begin{bmatrix} 4.90 & -0.40 & -11.92 \\ 5.78 & -1.6 & 4.43 \\ 4.09 & 9.69 & 9.36 \end{bmatrix}$	Q=diag(10,10,100) R=diag(.05,.05,.05) eigenvalues: 0.08, 0.30, 0.62 MO = 3; FB	±20% disturbance in feed flowrate (Figure 5.10)

* For explanation of Run No. and other notation see section on Notation, page 250.

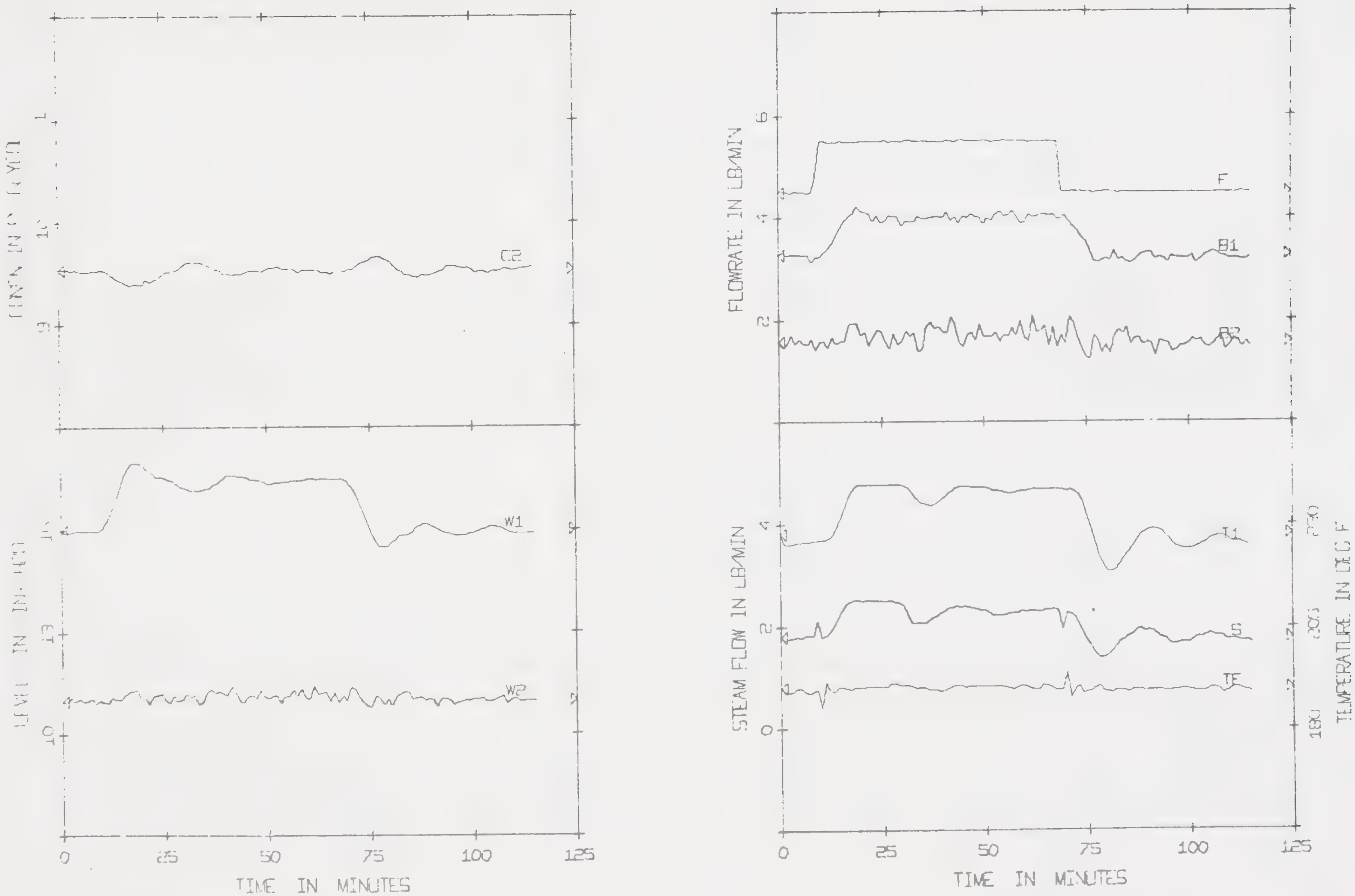

Figure 5.9: Experimental proportional feedback control (DMIN/01/P) (K_{dmin} (Table 5.4), MO/3, CO/3, ± 20% F)

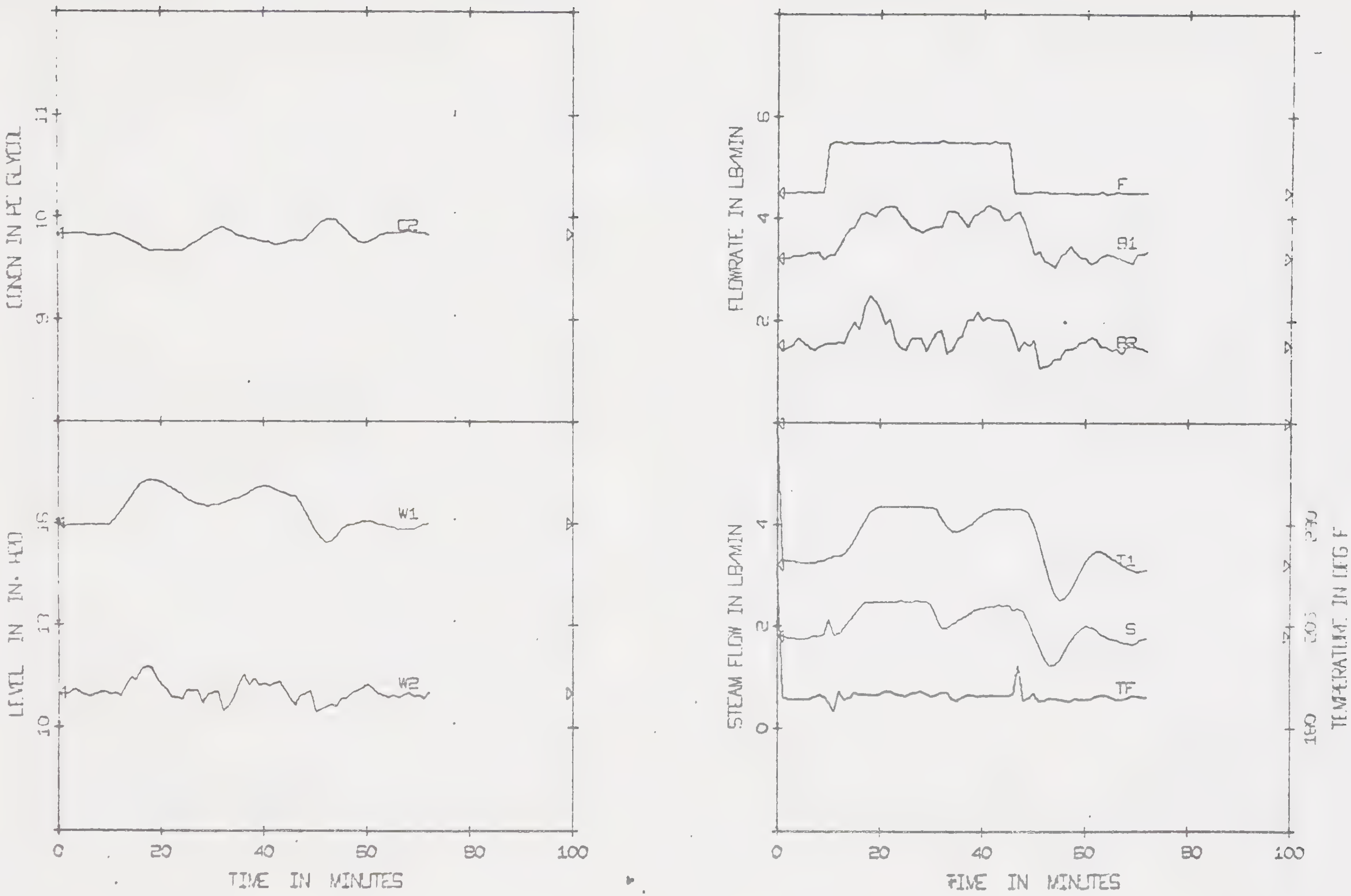

Figure 5.10: Experimental proportional feedback control (OPT/01/P)
(K_{opt} (Table 5.4), MO/3, CO/3, ± 20% F)

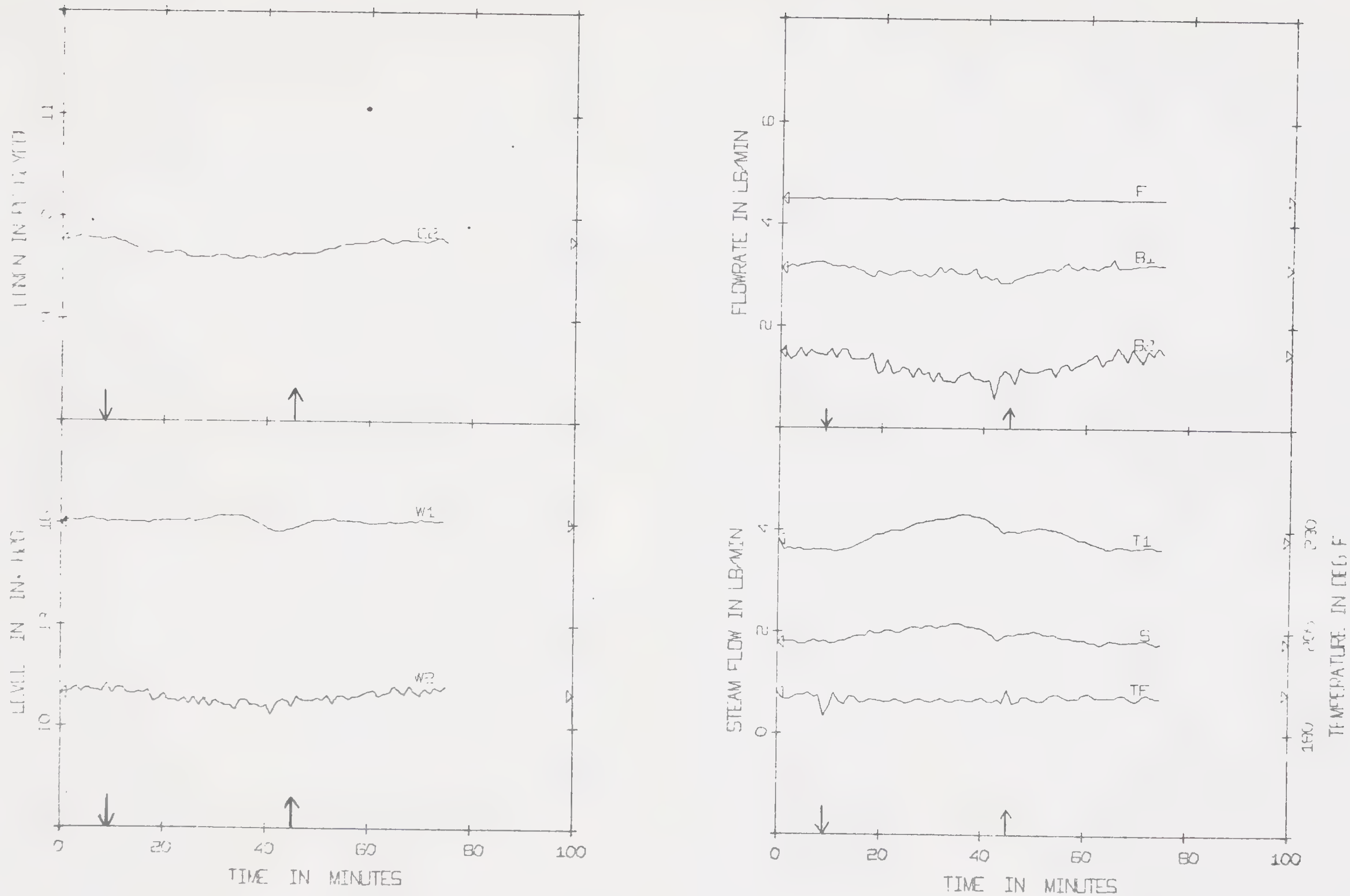

Figure 5.11: Experimental proportional feedback control (DMIN/01/P) (K_{dmin} (Table 5.4), MO/3, CO/3, ∓ 30% CF (at time indicated by arrows on t axis))

± 30% disturbance was introduced in the feed concentration, CF, with controller K_{dmin} implemented. Figure 5.11 shows the results of this run. Even when most of the control effort (via design) was concentrated in minimising the effect of F on C2, the product concentration sensitivity to CF had not increased.

5.8 Conclusions

i) The suggested procedure allows simultaneous eigenvalue assignment and disturbance minimisation and has immediate applications to the design of observers for systems with unmeasurable disturbances.

ii) The objective functional defined by eqn. (5.3) is more easily evaluated than in Willems, method [10]. The basic computational requirements for the design procedure are modest except for the numerical search procedure which does not always perform well, especially for higher order systems.

iii) The illustrative example shows the advantages of the proposed method in comparison to methods such as pole assignment by unity-rank state feedback which do not take into account the information available in the disturbance matrix, D, of the process models.

iv) An experimental evaluation of the proposed design scheme on the pilot-plant double effect evaporator gave good results and also demonstrated the practicality of the method.

Some aspects of the proposed procedure are worthy of further considerations. For example, it would be useful to investigate necessary and sufficient conditions for a minimum $J_L = 0$ to exist (at certain $\omega \neq 0$) for a class of unknown disturbances that satisfy a differential equation. Obviously the results in [1-3] are sufficient but not necessary for J_L to be identically zero.

CHAPTER 6

EIGENVALUE INVARIANCE TO SYSTEM PARAMETER VARIATIONS BY EIGENVECTOR ASSIGNMENT

Abstract:

A method is presented for designing constant multivariable feedback controllers that make selected closed-loop eigenvalues *invariant* to *unknown* perturbations of *arbitrary* magnitude in system parameters. The sufficient conditions for eigenvalue invariance are expressed in terms of the structure of the closed-loop system matrices, and a numerical example is included to show how eigenvalue/eigenvector assignment techniques can be used to produce the desired eigenvector structure and thereby, eigenvalue invariance.

6.1 Introduction

In most industrial applications, perturbations or errors in the system parameters and/or feedback gains are quite common. These perturbations or errors might be due to changes in the plant model, faulty instrumentation, etc. If the control system design is based on a specific set of assumed model parameters, then perturbations in these parameter values can cause deterioration in the plant performance, or may even result in an unstable system.

Eigenvalue or pole-placement algorithms have been widely used in both single variable and multivariable design applications because they give the user the ability to specify system stability and general characteristics of the time domain response. Hence it would be desirable when using pole placement (or other modern design techniques that rely on system models) to ensure, as much as possible, that perturbations in system parameters do not negate the original design objectives.

The purpose of this chapter is to show how, and under what conditions, selected closed-loop eigenvalues can be made invariant to perturbations in the system parameters. The question of how many, and which eigenvalues can be made invariant to perturbations is also considered.

This chapter is organised as follows. Section 6.2 discusses relevant previous work carried out in the area of eigenvalue insensitivity, Section 6.3 formulates the control problem and Section 6.4 is concerned with derivation of results to achieve eigenvalue insensitivity to *small* perturbations. In Sections 6.5 and 6.6 we consider the more important result, namely achieving eigenvalue invariance to *arbitrary* perturbations by eigenvector assignment, Section 6.7 deals with a numerical example to illustrate the design paper and is followed by Section 6.8 on Conclusions.

6.2 Previous Work

The problem of sensitivity reduction of closed-loop eigenvalues has been previously considered by Tzafestas and Paraskevopoulos [1], who designed a state feedback controller to assign closed-loop eigenvalues and to reduce eigenvalue sensitivity to parameter variations. More recently, Mita and Ngamkajornvivat [2] have outlined a method for single-input systems that achieve eigenvalue invariance to a class of unity-rank perturbations in the system matrix. Seraji [3] has proposed a method in which unity rank state feedback control is employed to assign closed-loop eigenvalues and also make the dominant eigenvalues insensitive to given variations

in the system parameters. This procedure is not suitable for programming on a computer, and also leads to various difficulties when applied to high order systems. Gourishankar and Ramar [4] have outlined methods for the design of unity and non-unity rank state feedback controllers which assign closed-loop eigenvalues and also minimise a measure of eigenvalue sensitivity to given parameter variations. Karlin et al. [5] have considered the synthesis of feedback control laws which make linear system state or output trajectories insensitive to small parameter variations. The existence of such control laws necessarily requires restrictive conditions.

6.3 Statement of the Problem

Consider the following n^{th} order, completely controllable, linear multivariable system with r inputs, u(t), q disturbances, $\xi(t)$, and m outputs, y(t):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + D\xi(t) \\ y(t) &= Cx(t) \end{aligned} \tag{6.1}$$

and the state feedback control law:

$$u(t) = Kx(t) \tag{6.2}$$

The resulting closed-loop system is described by:

$$\dot{x}(t) = Hx(t) + D\xi(t) \tag{6.3}$$

where $H = A + BK$.

In terms of individual matrix elements:

$$h_{ij} = a_{ij} + b_i^T k_j \tag{6.4}$$

where h_{ij} and a_{ij} represent the $(i,j)^{th}$ elements of H and A respectively; b_i^T represents the i^{th} row of B; and k_j represents the j^{th} column of K. Let the perturbations or variations in H, due to variations in A or K, be denoted by δH. In this investigation the main questions of interest are:

i) Under what conditions can the closed-loop eigenvalues be made invariant to perturbations δH?

ii) How many and which eigenvalues can be made invariant?

iii) What design procedure will produce eigenvalue invariance using only proportional feedback control?

6.3.1 Practical Significance

The case where the perturbations δH must be known, constant values is of little practical significance

because it would then usually be possible to carry out the design of the control system based on the corrected system matrix $(H + \delta H)$. However, the case where the perturbations are unknown is of considerable importance. For example, modelling errors and/or real-time variations can produce errors in the i^{th} column of A, which directly affect the i^{th} column of H. Variations in the transducer gains in the i^{th} loop, and any other perturbations that affect only the i^{th} column of K, can also be handled directly since they contribute only to the i^{th} column of H. Thus variations in the i^{th} column of H can be represented as:

$$\delta H = \begin{bmatrix} 0 \ldots \delta a_{1i} + b_1^T(\delta k_i) & \ldots & 0 \\ 0 \ldots \delta a_{2i} + b_2^T(\delta k_i) & \ldots & 0 \\ \vdots \qquad \vdots & & \vdots \\ 0 \ldots \delta a_{ni} + b_n^T(\delta k_i) & \ldots & 0 \end{bmatrix}$$

where b_i^T represents the i^{th} row of B.

Note that for the general case, variations in elements of B cannot be handled by the procedures presented in this paper. However, Ramar and Gourishankar [6] have recently proposed a method for the design of unrestricted rank state feedback that minimises the sensitivity of closed-loop poles to variations in matrices B or K. For special cases where the number of inputs is

equal to n then complete eigenvalue invariance to perturbations in B can be achieved by considering the dual or the adjoint of the system in eqn. (6.1) (Chapter 2, see also Shah et al. [7]).

6.4 Eigenvalue Insensitivity to Small Perturbations

Faddeeva and Faddeev [8] have shown that the following first-order relationship can be used to approximate the change in a distinct eigenvalue, λ_i, due to small perturbations in the elements of the system matrix H:

$$\delta\lambda_i = v_i^T(\delta H)\, w_i \tag{6.6}$$

where w_i and v_i are the i^{th} eigenvector and reciprocal eigenvector of H respectively. Therefore, for a given system the i^{th} eigenvalue will be invariant to small perturbations, if the right hand side of eqn. (6.6) is identically zero. The four conditions under which this can occur are:

$$\begin{aligned}
&\text{a)} \quad (\delta H)\, w_i = 0 \\
&\text{b)} \quad v_i^T\, (\delta H) = 0 \\
&\text{c)} \quad (\delta H)\, w_i = \alpha w_j \text{ where } i \neq j \text{ and } \alpha \text{ is a non-zero scalar} \\
&\text{d)} \quad v_i^T\, (\delta H) = \beta v_j^T \text{ where } i \neq j \text{ and } \beta \text{ is a non-zero scalar .}
\end{aligned} \tag{6.7}$$

6.4.1 Eigenvalue Insensitivity by Eigenvector Assignment

The approach taken first in the following analysis is to examine the system eigenvectors and determine under what conditions eqn. (6.7a) is satisfied.

Obviously, if

$$\{ w_i \} \subset \mathrm{Ker}\{\delta H\} \qquad i = 1, \ldots, p \quad (6.8)$$

where p = dimension Ker{δH} then eqn. (6.7a) is satisfied and at least the p closed-loop eigenvalues corresponding to the eigenvectors $\{w_i, i = 1, \ldots, p\}$ are insensitive to any small perturbations, δH.

One approach to the design problem would be to specify the closed-loop eigenvectors such that they satisfy condition (6.8) directly. Unfortunately when constant state feedback control is used to assign closed-loop eigenvalues and eigenvectors, then only r elements (where r is the number of inputs) of each of the n eigenvectors can be assigned arbitrarily, and this is possible if and only if:

i) the resulting n eigenvectors are linearly independent, and

ii) the corresponding r rows of the input matrix B are linearly independent.

(cf. Chapter 2, and [7]).

Because (n-r) elements of each eigenvector cannot be assigned arbitrarily, and because it is important to distinguish between arbitrary and known system perturbations, it is convenient to reorder the columns of δH (if necessary) and partition δH such that:

$$\delta H = \{\delta H_1 \,\vdots\, \delta H_2 \,\vdots\, \delta H_3\} \tag{6.9}$$

where δH_1 is an nxk matrix containing all the unknown perturbations ($k \leq r$)

δH_2 is an nx(r-k) matrix of known constant perturbations

δH_3 is an nx(n-r) matrix of known constant perturbations.

Then, if the i^{th} eigenvector is partitioned into an rxl vector $W1_i$ and a (n-r)xl vector, $W2_i$, eqn. (6.7a) can be rewritten as:

$$[\delta H_1 \,\vdots\, \delta H_2 \,\vdots\, \delta H_3] \begin{bmatrix} W1_i \\ \hline W2_i \end{bmatrix} = 0 \tag{6.10}$$

Since the r elements of $W1_i$ can be assigned arbitrarily, it is usually possible (subject to the two constraints noted above) to satisfy the condition:

$$[\delta H_1 \,\vdots\, \delta H_2]\ W1_i = 0 \quad \text{for } i = 1, \ldots, p \tag{6.11}$$

However, the second part of the sufficiency condition in eqn. (6.10) requires that:

$$\delta H_3 \, W2_i = 0 \quad \text{for} \quad i = 1, \ldots, p \tag{6.12}$$

and $W2_i$ cannot be specified arbitrarily. Sufficient conditions for eqn. (6.12) to hold are:

$$\delta H_3 \, A_3 = 0$$

$$\delta H_3 \, A_4 = 0 \tag{6.13}$$

and

$$\delta H_3 \, B_2 = 0$$

where A_3, A_4 and B_2 are partitions of A and B as shown in the following partitioned form of H = A + BK = WJV:

$$\left[\begin{array}{c|c} H_1 & H_2 \\ \hline H_3 & H_4 \end{array}\right] = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] [K_1 \;\; K_2] = \left[\begin{array}{c|c} W_1 & W_2 \\ \hline W_3 & W_4 \end{array}\right] \left[\begin{array}{c|c} J_1 & 0 \\ \hline 0 & J_2 \end{array}\right] \left[\begin{array}{c|c} V_1 & V_2 \\ \hline V_3 & V_4 \end{array}\right] \tag{6.14}$$

where A_1, B_1, K_1 and $H_1 \varepsilon \mathcal{R}^{rxr}$; A_4 and $H_4 \varepsilon \mathcal{R}^{(n-r)x(n-r)}$; $B_2 \varepsilon \mathcal{R}^{(n-r)xr}$ matrix; $K_2 \varepsilon \mathcal{R}^{rx(n-r)}$; and W, J and $V = W^{-1}$, have also been appropriately partitioned.

Proof for Sufficiency

The sufficient conditions for eqn. (6.12) to hold, as given by eqn. (6.13), can be derived starting with the following two equations which are a consequence of eqn. (6.14):

$$\begin{aligned} W_3J_1V_1 + W_4J_2V_3 &= A_3 + B_2K_1 \\ W_3J_1V_2 + W_4J_2V_4 &= A_4 + B_2K_2 \end{aligned} \tag{6.15}$$

Assuming eqn. (6.13) to be true, then pre-multiplying both of eqns. (6.15) by δH_3; post-multiplying the first equation in (6.15) by W_1 and the second by W_3; and then adding the two equations yields:

$$\delta H_3W_3J_1(V_1W_1 + V_2W_3) + \delta H_3W_4J_2(V_3W_1 + V_4W_3) = 0 \tag{6.16}$$

From the identity $VW = I$ it is known that $V_1W_1 + V_2W_3 = I_r$ and $V_3W_1 + V_4W_3 = 0$ and hence, knowing that $J_1 \neq 0$, eqn. (6.16) implies

$$\delta H_3W_3 = 0 \tag{6.17}$$

Likewise pre-multiplying both equations in (6.15) by δH_3; post-multiplying the first equation by W_2 and the second by W_4; and then adding the two equations yields:

$$\delta H_3 W_3 J_1 (V_1 W_2 + V_2 W_4) + \delta H_3 W_4 J_2 (V_3 W_2 + V_4 W_4) = 0 \tag{6.18}$$

By the use of the identity VW = I, eqn. (6.18) implies:

$$\delta H_3 W_4 = 0 \tag{6.19}$$

since,

$$[W2_1 \; W2_2 \; .... \; W2_i \; .... \; W2_n] = [W_3 \; W_4]$$

eqns. (6.17) and (6.19) show that eqn. (6.12) holds if the conditions expressed by eqn. (6.13) are correct. This proves sufficiency.

The sufficient conditions for eigenvalue invariance expressed in eqn. (6.11) and (6.13) are obviously very restrictive but it should be noted that they are satisfied for the special case when $\delta H_3 = 0$, i.e. when there are no variations in at least (n-r) columns of H. In this case at least p eigenvalues can be made invariant where;

$$p = n\text{-rank } \{\delta H_1\} - \text{rank } \{\delta H_2\} \tag{6.20}$$

Normally, since all k columns of δH_1 contain arbitrary elements, it can be assumed that rank $\{\delta H_1\} = k$, and that its columns are linearly independent with those of δH_2.

The above development of conditions governing eigenvalue invariance to small perturbations in H was based on eqn. (6.7a). However, it can easily be shown that if eqn. (6.7a) is satisfied then the use of eqn. (6.7b) will not lead to any additional invariant eigenvalues. This follows since eqn. (6.8) implies that rank $(\delta H) = n-p$ and that the eigenvectors $\{w_i, i = p+1, \ldots, n\}$ span δH. Therefore $v_i^T(\delta H) \neq 0$ for $i = p+1, \ldots, n$, i.e. if condition (6.7a) is satisfied, condition (6.7b) cannot be met for $i>p$. Also it was not possible to develop an improved design procedure based on eqns. (6.7c) and (6.7d) and hence they are not considered further in this paper.

The results of the above discussion can be summarized as follows. Sufficient conditions for $(n-k)$ eigenvalues of the closed-loop system to remain insensitive to unknown perturbations in elements of H are:

i) that the perturbations be small,

ii) that the perturbations occur in only the first k columns where k is less than or equal to the number of inputs, r,

iii) that the elements in the remaining $(n-k)$ columns of H be known constant values,

iv) the first k elements of each of the $(n-k)$ eigenvectors $\{w_i, i = k+1, \ldots, n\}$ be zero.

In the following sections, sufficient conditions under which eigenvalues can be made invariant to *unknown* perturbations, δH, of arbitrarily large magnitude are developed and a design procedure is presented that will produce a closed-loop system that meets these conditions.

6.5 Eigenvalue Invariance to Arbitrary Perturbations

The following sufficient conditions for eigenvalue invariance are derived based on the properties of quasi-triangular matrices rather than starting with eqn. (6.6) which is only a first order approximation.

THEOREM 6.1:

A sufficient condition for (n-k) of the eigenvalues of the closed-loop system matrix, H, to remain invariant to arbitrary and unknown perturbations in the first k columns of H, is that H be of the following form, or that it can be brought into the following form by reordering of the state variables:

$$H = \begin{bmatrix} H_1 & 0 \\ H_3 & H_4 \end{bmatrix} \qquad (6.21)$$

where $H_1 \in R^{k x k}$, $H_4 \in R^{(n-k) x (n-k)}$; *the (n-k) invariant eigenvalues* $\{\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n\}$ *are the eigenvalues*

of H_4 and all perturbations are assumed to occur in H_1 and/or H_3.

Proof of Theorem 6.1:

If H is quasi-triangular as defined by eqn. (6.21), then the n eigenvalues of H are the k eigenvalues of H_1 and the (n-k) eigenvalues of H_4. By assumption the perturbations occur only in H_1 and/or H_3 and hence H_4, and its eigenvalues are invariant. Alternatively, from eqn. (6.14) it follows that $H_2 = A_2+B_1K_2$ and $H_4 = A_4+B_2K_2$ and hence are invariant to perturbations in A_1, A_3 and K_1.

Remark 6.1: The sufficient conditions holds for $k \leq n$ but design using eigenvector assignment requires $k \leq r$.

Remark 6.2: If $H_2 = 0$ then the eigenvalues of H are the eigenvalues of H_1 plus H_4, which are both independent of A_3. Therefore all of the n closed-loop eigenvalues are invariant to perturbations in A_3.

Remark 6.3: The time domain trajectories of the state variables $\{x_i, i = k+1, \ldots n\}$ due to initial condition or disturbance excitation are invariant to perturbations in the first k columns of H if the theorem is satisfied and the initial conditions and the disturbance matrix, D, satisfy the conditions:

$$[H_1 \,\vdots\, H_2]x(0) = 0$$

$$[H_1 \,\vdots\, H_2]D = 0 \qquad (6.22)$$

where $H_2 = 0$

Cruz and Perkins [9] have also derived a similar result. However, their result is not constructive, it is essentially a structural result.

Remark 6.4: If the output matrix C is of the form $C = [I_k \,\vdots\, 0]$ then the system modes associated with the invariant eigenvalues are unobservable. Furthermore, for such a C, the invariant zeros defined and discussed by Bengtsson [10] are the same as some (or all) of the invariant eigenvalues as defined by Theorem 6.1. The relationship between invariant zeros and eigenvalue invariance is explored in detail in Appendix D.

Remark 6.5: An H matrix which satisfies the conditions of Theorem 6.1, also satisfies the sufficient conditions for insensitivity to small perturbations discussed in Section 6.2. However, Theorem 6.1 provides sufficient conditions for invariance to perturbations of *arbitrary* magnitude.

As shown in the next section and by the numerical example, it is possible to design state feedback controllers that produce a closed-loop system matrix, H, with the

structure defined in eqn. (6.21), and hence invariant eigenvalues.

6.6 Design Procedure for Eigenvalue Invariance

It has been shown in Chapter 3 that if $H_2 = 0$ then $W_2 = 0$ and vice-versa (see also Shah et al. [11]). Therefore an alternative condition to that expressed in Theorem 6.1 is that W have the structure:

$$W = \left[\begin{array}{c|c} W_1 & 0 \\ \hline W_3 & W_4 \end{array}\right] \tag{6.23}$$

Thus any eigenvector assignment procedure that sets $W_2 = 0$ will result in a system with (n-k) eigenvalues that are invariant to perturbations in the first k columns of H. The algorithm developed by Srinathkumar and Rhoten [12], discussed in Chapters 2 and 3, permits arbitrary assignment of up to r elements of each eigenvector plus all of the system eigenvalues and thus can be used to achieve eigenvalue placement and invariance. The main restrictions are:

i) The eigenvalues must be distinct; $V = W^{-1}$ must exist; and B_1 (cf eqn. 6.14) must be non-singular as noted earlier.

ii) If k, the number of columns of H containing arbitrary and unknown perturbations, is less than the number of inputs, r, then W_2 can be set to zero and all eigenvalues can be assigned arbitrarily. However, if k = r then (n-r) eigenvalues of the closed-loop system are the eigenvalues of $[A_4 - B_2B_1^{-1}A_2]$ and cannot be specified arbitrarily (cf. Chapter 3).

6.7 A Numerical Example

A linear, third-order system with two inputs (Gourishankar and Ramar, [4]) is used to illustrate the design procedure for eigenvalue invariance. The system is open-loop, unstable with eigenvalues at 0, 1 and 1 and coefficient matrices (cf. eqn. 6.1):

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (6 \cdot 24)$$

For the purpose of this paper it is assumed that there are errors in the transducer gains associated with x_2 plus perturbations in the second column of A. The design objective is to place the eigenvalues at -1, -2 and -3

respectively and also to make the dominant eigenvalues invariant to the errors associated with x_2 and to the parameter variations in A.

The first step is to reorder the state vector such that $\tilde{x}^T = [x_2, x_1, x_3]$ and hence that all of the arbitrary and unknown variations are in the first column of $\tilde{H}$. (The superscript $\tilde{}$ denotes the reordered system). Then eqn. (6.20) indicates that at least p = 3-1-0 = 2 eigenvalues can be made invariant. The design procedure for eigenvalue/eigenvector assignment using constant state feedback (Srinathkumar and Rhoten [12]) can then be used to assign the eigenvalues to -3, -2, -1 and the first two elements of each eigenvector as (cf. eqn. (6.14)).

$$\tilde{W}_1 = \begin{bmatrix} 0.1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \tilde{W}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6.25}$$

The resulting closed-loop eigenvector and system matrices are:

$$\tilde{W} = \left[\begin{array}{c|cc} 0.1 & 0 & 0 \\ \hline 1 & 1 & 1 \\ .775 & .667 & 0.5 \end{array}\right] \qquad \tilde{H} = \left[\begin{array}{c|cc} -3.0 & 0 & 0 \\ \hline -3.5 & 2 & -6 \\ -4.5 & 2 & -5 \end{array}\right] \tag{6.26}$$

The corresponding feedback control matrix calculated from the relationship $H = A + BK$, using the pseudoinverse of B is denoted by K1 and is shown in Table 6.1. It is obvious that $\tilde{W}$, and hence $\tilde{H}$, in eqn. (6.26) have the structure specified by Theorem 6.1 with k=1. Hence, Theorem 6.1 shows that the (n-k) eigenvalues -2 and -1 will be invariant to arbitrary perturbations in the first k=1 columns of $\tilde{H}$. (Note the following discussion is in terms of the original rather than the reordered system.)

Table 6.1 includes the numerical values for $\partial\lambda_i/\partial h_{j2}$, i.e. the sensitivity of the i^{th} eigenvalue to variations in the j^{th} element of the second column of H. These values confirm that eigenvalue/eigenvector assignment design procedure did make the two dominant eigenvalues, -1 and -2, invariant to arbitrary variations in the second column of H. For comparison purposes three other feedback matrices that position the closed-loop eigenvalues at -1, -2 and -3 were also considered. The results of Gourishankar and Ramar [4], who employed a numerical minimisation approach to reduce eigenvalue sensitivity to variations in h_{12} and h_{22} for the same example, are included in Table 6.1 where K2 denotes their state feedback matrix. Table 1 also includes results for two unity-rank feedback matrices, K3 and K4, designed using conventional pole assignment techniques.

The sensitivity of the eigenvalues of the closed-

loop system using K2 as the state feedback controller (Gourishankar and Ramar [4]) is small as would be expected from using a minimisation procedure. Note, however, that the elements of K2 are generally larger than those of K1. The proposed eigenvector design method is relatively straightforward and requires much less computational effort than does the minimisation procedure. Furthermore, the proposed method avoids the convergence and dimensionality problems encountered in minimisation procedures such as the one used in [4].

It is well known that when a state feedback matrix, K, is constrained to have the unity rank structure, $K = gf^T$, the choice of the rx1 vector, g, is arbitrary as long as the single-input system (A, Bg) is controllable. In computing K3 the choice, $g = [1\ 1]^T$, was made. For this controller the sensitivity of the eigenvalues to variations in the second column of H was found to be high (see row 3 of Table 6·1). Different values of g were then selected by trial and error and it was found that for $g = [1\ 0]^T$, the resulting controller, K4, had reduced eigenvalue sensitivity to variations in the second column of H. The point to note here is that most conventional pole placement design techniques give little guidance as to how g should be selected so as to reduce eigenvalue sensitivity.

The significance of the numerical values of the eigenvalue sensitivity coefficients is illustrated by the

simulated responses in Fig. 6.1. Here, the closed-loop response of x_2 to the initial condition, $x_2(0) = 1.0$ is shown for each of the four controllers (solid line). Also included in the figure is the closed-loop response when a 20% error in x_2 exists due to variations in the transducer gain (dashed curves). The figure clearly illustrates the advantages of using an eigenvalue invariance, or a minimum eigenvalue sensitivity procedure, when designing for pole placement.

The results are not reproduced here, but the time-domain trajectories of x_1 and x_3 (using K_1) to initial conditions excitation of $x_1(0) = 1.0$ and $x_3(0) = 1.0$ were completely invariant to the errors in x_2 since the system meets the conditions of Theorem 6.1 and Remark 6.3.

Note that Theorem 6.1 means that even if the controller design is based on a model that contains errors in the first k columns of A (or of H), then n-k eigenvalues will still retain their assigned values and be invariant. In this example, as implied by Remark 6.2, <u>all</u> of the system eigenvalues are completely invariant to arbitrary variations in $\tilde{A}_3$, i.e. in $\tilde{a}_{21}$ and $\tilde{a}_{31}$. Variations in $\tilde{A}_1 = \tilde{a}_{11}$ would affect only the eigenvalues of $\tilde{H}_1$.

6.8 Conclusions

Theorem 6.1 provides sufficient conditions for invariance of (n-k) eigenvalues of a closed-loop system to arbitrary and unknown perturbations in k columns of the closed-loop system matrix, H. By contrast, most previous work is based on first-order approximations to the variations in closed-loop eigenvalues and provides only *insensitivity* to *small* and/or known perturbations. However, the important result from a design point of view is that in many practical applications, selected eigenvalues can be made invariant to factors such as poor parameter estimates, modelling errors, variations in transducer gains, etc. The illustrative example shows the advantages of the eigenvalue/eigenvector assignment technique in comparison to other methods for reducing the effect of parameter variations on selected closed-loop eigenvalues.

TABLE 6.1: Details of Eigenvalue Insensitive Controllers For the Illustrative Example of Section 6.7

Closed-loop eigenvalues: $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = -3$

DESIGN METHOD**	State feedback matrix, K	Value of j	$\frac{\partial\lambda_1}{\partial h_{j2}}$ *	$\frac{\partial\lambda_2}{\partial h_{j2}}$ *	$\frac{\partial\lambda_3}{\partial h_{j2}}$ *
1	$K1 = \begin{bmatrix} 2.0 & -4.502 & -6.0 \\ 0 & -4.0 & -1.0 \end{bmatrix}$	1 2 3	0 0 0	0 0 0	0 1.0 0
2	$K2 = \begin{bmatrix} 3.614 & 10.0 & -8.590 \\ -.167 & -3.02 & -0.751 \end{bmatrix}$	1 2 3	-0.100 0.317 0.143	0.034 0.346 -0.036	0.066 0.337 -0.107
3	$K3 = \begin{bmatrix} -6 & -18 & 16 \\ -6 & -18 & 16 \end{bmatrix}$	1 2 3	3.005 3.013 -5.514	-12.021 -20.061 28.065	9.016 18.048 -22.551
4	$K4 = \begin{bmatrix} -3 & -21 & -5 \\ 0 & 0 & 0 \end{bmatrix}$	1 2 3	-1.505 -4.521 1.002	3.012 13.051 -1.005	-1.507 -7.529 0.003

**Key

1 eigenvalue/eigenvector assignment
2 eigenvalue assignment with minimum sensitivity (Gourishankar and Ramar, [4])
3 eigenvalue assignment, $K3 = gf^T$, $g = [1\ 1]^T$
4 eigenvalue assignment, $K4 = gf^T$, $g = [1\ 0]^T$

* $\frac{\partial\lambda_i}{\partial h_i}$ denotes the sensitivity or variation of λ_i with respect to a small variation in the (j,2) element of H.

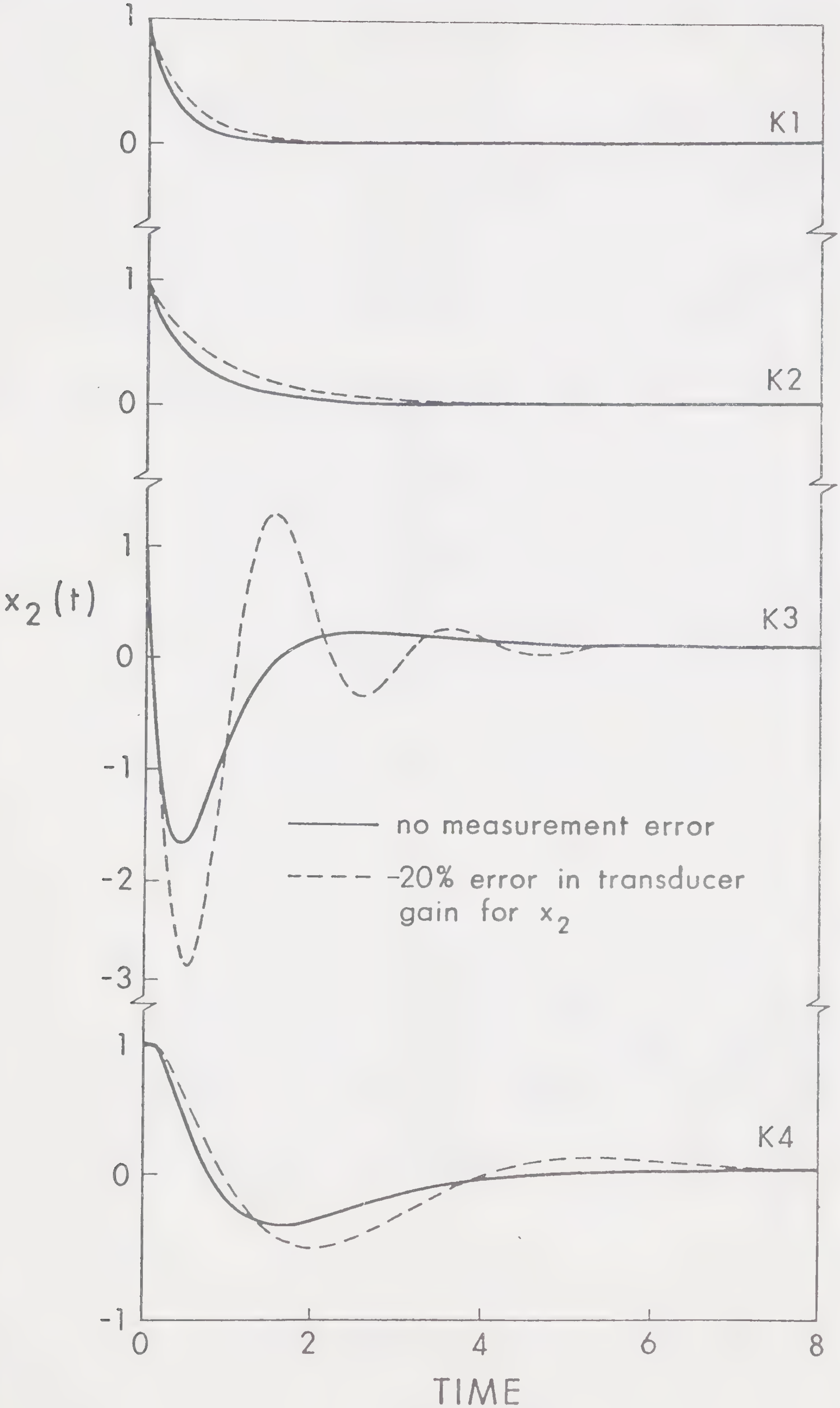

FIGURE 6.1: Simulated closed-loop responses of the state space model defined by equation (6.24) and the four proportional feedback controllers in Table 6.1.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

7.1 Contribution of This Thesis

The main contributions of this study are:

i) Derivation of necessary and sufficient conditions for undisturbability, and the use of these and related results is the basis for the development of a simple systematic, design procedure;

ii) A basis for direct analysis of the structure of systems which is easy to apply, and is useful from a control system design point of view;

iii) A detailed evaluation of the new design technique, by experimental application to the double effect evaporator, and by digital simulation of 11^{th} and 20^{th} order models of two different binary distillation columns.

iv) Development of a new procedure for designing state feedback controllers to assign closed-loop system eigenvalues and also minimise the effect of external disturbances on system outputs of interest;

v) Development of a simple and constructive result for designing constant multivariable feedback controllers to make selected closed-loop eigenvalues invariant to arbitrary variations in system parameters;

vi) An algorithm to compute the invariant zeros of a system.

7.2 Conclusions

In this thesis the role of eigenvalues and eigenvectors in multivariable control systems design has been illustrated by relating the properties of undisturbability and eigenvalue invariance to the structure of the open or closed-loop system eigenvector matrix.

The concept of undisturbability has been formally defined and related to such concepts as uncontrollability, non-accessibility, and structural controllability. Necessary and sufficient conditions for undisturbability

have been expressed in terms of both the structure of the coefficient matrices of the state-space model and also in terms of the structure of the system eigenvector matrix. These results provide new insight into the structure of linear multivariable systems and provide a basis for the analysis and synthesis of regulatory controllers. Specifically, the results allow determination of the property of undisturbability of specific input-output pairs by a simple inspection of the zero entries in the appropriate matrices. Existing design procedures for eigenvalue/eigenvector assignment have been reviewed and conditions for eigenvalue/eigenvector assignment have been discussed here in terms of range space restrictions. The results on undisturbability, together with an existing algorithm for eigenvalue/eigenvector assignment lead to a simple and constructive design procedure for synthesizing multivariable controllers which provide disturbance localisation. An important characteristic of this design procedure is that it makes use of the information contained in the model about the effect of disturbances, whereas many other design schemes for feedback controllers do not.

Experimental evaluation of feedback plus feedforward controllers designed to produce undisturbability was carried out by application to a computer controlled,

pilot-plant evaporator. The design procedure was further evaluated by digital simulation of 11^{th} and 20^{th} order models of two different distillation columns. These applications demonstrate the simplicity of the design procedure in its application to large order systems. The problems of simultaneous disturbance localisation with asymptotic setpoint tracking, and disturbance localisation using proportional plus integral feedback have also been investigated.

A design procedure for minimising the effect of external disturbances on specific system outputs of interest and assigning eigenvalues in a closed-loop system, has been developed. This method can also be used in the design of full order observers for systems with unmeasurable disturbances. Experimental evaluation of the proposed minimisation scheme on the double effect evaporator gave good results and also demonstrated the practicality of the method.

Simple constructive results for the design of constant multivariable feedback controllers, to make selected closed-loop eigenvalues invariant to unknown perturbations of arbitrary magnitude in system parameters, are given. In general, these results show the advantages of having a quasi-triangular (or decomposed)

system structure either for undisturbability, or for making dominant eigenvalues invariant to arbitrary perturbations in specified system parameters. A discussion on the relationship between such concepts as invariant zeros, (A,B)-invariant subspaces, and parametric sensitivity is also included. One outcome of this discussion has been a simple algorithm for computing the invariant zeros of a system.

7.3 Recommendations for Future Work

As a result of work done in this study, several new research areas worthy of further consideration have been identified. Some of these are:

7.3.1 Structural observability

Lin [1] first introduced the idea of structural controllability in 1974. Since then Shields and Pearson [2], and Glover and Silverman [3] have extended this concept to also apply to multi-input systems. The duality of these results for investigating 'structural' observability has not been formally established. Because of the algebraic nature of the results on undisturbability, duality between undisturbability and unobservability is not difficult to establish. When a one to one

correspondence between undisturbable modes and state variables exists, the undisturbable modes in a dual or adjoint system are the unobservable modes or state variables. In fact, the discussion on undisturbability and structural controllability can be immediately translated by using duality to relate undisturbability to 'structural' unobservability. However, the limitation here is that the dual of undisturbable subspace is smaller than or equal to the 'structurally' unobservable subspace, and therefore the complete class of structurally unobservable subspaces cannot be identified.

It is not difficult to see that the concept of structural observability can be of great use in sensor location and measurement studies in the plant design stage (see remarks on the control scheme for the 20th order distillation column - Section 4.5.2). Structural observability considerations can also influence sensor location and measurement in systems that require state estimation or filtering, i.e. the object here would be to locate sensors such that the '(structural) observability index' is maximised.

7.3.2. Use of structural analysis in system design

Determination of structural disturbability,

controllability (and 'observability') in a system requires very little information - i.e. only knowledge of the zero elements in the system. As a result such an analysis prior to the actual plant design and construction stage can be of help in avoiding potential pitfalls (e.g. in avoiding ineffective control inputs), or it can help identify and suggest modifications that may help the final control of the plant (e.g. in modifying the feed system to make a critical plant process undisturbable). Such an analytical technique needs to be systematised for use as an effective design tool.

It is not difficult to see that the structural results on undisturbability etc. also apply to graph or node related structures, for example computer flow diagrams or critical path schedules. Consequently, it would be worthwhile to investigate the application and relation of these concepts to such areas as the design and development of large fault-tolerant ($\equiv$ undisturbable) computer programs, or in the planning of critical path scheduling in face of uncertain tasks.

7.3.3 Disturbance minimisation as a design tool in the frequency domain context

The absolute magnitude of the transmittance

between a disturbance and an output has been used as a measure of disturbance/output coupling in the design procedure for disturbance minimisation in Chapter 5. Extension of this idea in the frequency domain context would consist of plotting a Nyquist array of the load transfer function matrix, $G_L(s)$, and then computing a dynamic or constant precompensator, K(s), to minimise the effect of disturbances on specific outputs at a specific frequency range of interest. There is a close parallel between the approach suggested here and the procedure due to Hawkins [4] which is concerned with pseudo-diagonalisation for the Inverse Nyquist array method of Rosenbrock [5].

Obviously the final design procedure would involve working simultaneously with Nyquist arrays of the process and the load transfer function matrices, G(s) and $G_L(s)$, respectively. Even though the computational requirements for such a procedure are significant (i.e. it would require use of an interactive computer graphics facility) there seems to be considerable scope for a design procedure that simultaneously considers the two important design requirements: regulation and setpoint following through a realisable controller, K(s).

7.3.4 Eigenvalue/eigenvector assignment using output feedback

In this thesis some of the difficulties associated with eigenvalue/eigenvector assignment using output feedback control have been identified. It appears from the work carried out here that such a direct procedure is prohibitively complex. An alternative procedure worthy of further consideration is the control law reduction techniques of Wilson [6] and Bengtsson [7] or the modification of these methods such that the reduced order output feedback control law guarantees system stability and also approximately retains the favourable eigenvector structure of the closed-loop system matrix (with state feedback). Development of new or modified control law reduction techniques, that would preserve desired structural characteristics is a worthwhile area for future work.

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NOTATION

General:

The field of real numbers and the field of complex numbers are denoted by $\mathcal{R}$ and $\mathcal{C}$ respectively. The n-dimensional vector space over the real numbers is written $\mathcal{R}^n$, and over the complex numbers $\mathcal{C}^n$. Upper case letters, e.g. A,B,C,D are used for matrices (linear maps). The range space of a matrix D is denoted by the corresponding script letter $\mathcal{D}$ (or $\mathcal{D} \equiv$ Image D). Script letters, e.g., $\mathcal{V}$, $\mathcal{W}$ are also used to denote linear subspaces. The distinction between range spaces and subspaces is clear from the context. The null space of a matrix C is written as Kernel (C) or sometimes as Ker(C). Lower case letters with or without subscripts, e.g. b_i, d_j, ℓ_i, denote vectors. Sometimes lower case letters also denote scalars. The distinction between vectors and scalars is clear from the context. The symbol, +, will be used for summation of linear subspaces, and the symbol $\oplus$, for the direct sum. $\sigma(\cdot)$ is used to denote the spectrum or eigenvalues of the matrix (argument). $(A/\mathcal{W})$ denotes the matrix representation of the restriction of A to $\mathcal{W}$ and $\mathcal{X}/\mathcal{W}$ denotes the quotient space of $\mathcal{X}$ modulo $\mathcal{W}$.

Specific

A	state coefficient matrix
B	control or input coefficient matrix
B_1, B_2	partitions of B
B^*	pseudoinverse of B
C	output coefficient matrix
D	disturbance or load coefficient matrix
E	partition of a transformation matrix
$G(s)$	transfer function matrix
H	closed-loop system matrix
I	identity matrix
J	canonical state matrix
K	state feedback control matrix
K^{FF}	feedforward control matrix
k	number of undisturbable state variables
L	disturbance (or load) coefficient matrix for the closed-loop system with feedforward control
ℓ_j	j^{th} column of L
m	dimension of output vector
n	dimension of state vector
q	dimension of disturbance vector
r	dimension of control vector
s	Laplace operator

Specific (continued)

u	control vector, rxl
V	closed-loop system reciprocal (or left) eigenvector matrix ($V=W^{-1}$)
W	closed-loop system eigenvector matrix
x	state vector, nxl
x^{ξ}	state response of the closed-loop system to disturbance
y	output vector mxl
y^{ξ}	output response of the closed-loop system to disturbance

Greek Letters

ϕ	state coefficient matrix for the discrete system
Δ	control coefficient matrix for the discrete system
θ	disturbance coefficient matrix for the discrete system
Λ	nxn diagonal matrix whose diagonal elements are the eigenvalues of H
ξ	disturbance vector, qxl

Superscripts

FF	feedforward
-1	matrix inverse
T	transpose of a vector or matrix
†	transform of a matrix
*	pseudoinverse of a matrix
1,2,3,4 or 11,12,21,22	refer to partitions of a matrix

Subscripts

i	component of a vector or row of a matrix
j	component of a vector or column of a matrix

Notation for Table 4.1 and Computer Plots of Evaporator Response

Each figure caption includes a string of characters delimited by parenthesis and/or semicolons which completely define the design and application conditions.

The general form and order of this information is as follows:

(run number)

(controller type (Table identification); model order; control type; run conditions).

The codes used for each of these specifications are defined in the following sections:

Run number

DL	-	disturbance localisation run
ML	-	multiloop run
OPT	-	optimal control run
EA	-	eigenvalue assignment run
MC	-	modal control run
P	-	proportional feedback control
P+FF	-	proportional feedback plus feedforward control
P+SP	-	proportional feedback plus setpoint control

Controller type

Subscript

$d\ell$	-	disturbance localisation controller
$m\ell$	-	multiloop controller
opt	-	optimal controller

Superscript

FF	-	feedforward controller
SP	-	setpoint controller

Model order

MO = 3	signifies controller design based on the 3^{rd} order model
MO = 5	signifies controller design based on the 5^{th} order model

Control type

FB - unless specified this denotes state feedback

FF - feedforward control of feed flowrate

SP - setpoint control

Run conditions

∓30% CF - denotes -30% step in CF followed by a +30% step in CF.

±20% F - denotes +20% step in F followed by a -20% step in F.

APPENDIX A

PRELIMINARIES

A.1 Matrix Pseudoinverse (cf. Greville [A.1]):

The matrix inverse is defined for square matrices only; it is used in the solution of sets of linear equations of the form $Mx=y$, so that $x=M^{-1}y$. In such linear equations the number of unknowns is equal to the number of equations. For nonsquare matrices used to describe systems of equations where the number of equations is not equal to the number of unknowns, the equivalent operator is the pseudoinverse. If a matrix B has more rows than columns, the pseudoinverse is defined as

$$B^* = (B^T B)^{-1} B^T$$

provided $(B^T B)$ is nonsingular. In the solution of linear equations, the set of equations, $Bx=y$ are the so-called overdetermined case - where there are more equations than unknowns. The resulting solution $x=B^* y$ is best in a least-squares sense. If a matrix, say C has more columns than rows, its pseudoinverse is then defined as:

$$C^{*} = C^{T}(CC^{T})^{-1}$$

This corresponds to the undetermined case - there are fewer equations than unknowns.

A.2 A-Invariant Subspace (Wonham and Morse [A.2])

A subspace $\mathcal{V} \subset \mathcal{X}$ is said to be *A-invariant* if $v \in \mathcal{V} \Rightarrow Av \in \mathcal{V}$, i.e. $A\mathcal{V} \subset \mathcal{V}$. The restricted map $\bar{A}:\mathcal{V}\to\mathcal{V}$ is defined so that $\bar{A}v = Av$ for all $v \in \mathcal{V}$. The restriction of A to $\mathcal{V}$ is denoted by $A/\mathcal{V}$. Some examples of A-invariant subspaces are:

i) If w_i is the i^{th} eigenvector of A, i.e. $A_{wi} = \lambda_i w_i$, then clearly the subspace $\mathcal{V} = \{w_1, w_2, \ldots., w_i\}$ is A-invariant. Furthermore if $w_1, w_2, \ldots, w_i$ are linearly independent then dimension $(\mathcal{V}) = i$.

ii) Let $\mathcal{R}_o$ denote the subspace of controllable states. $\mathcal{R}_o$ is given by:

$$\mathcal{R}_o \equiv \mathcal{B} + A\mathcal{B} + A^2\mathcal{B} + \ldots. + A^{n-1}\mathcal{B}$$

By Caley-Hamilton, $A\mathcal{R}_o \subset \mathcal{R}_o$, i.e. $\mathcal{R}_o$ is A-invariant.

iii) If θ denotes the subspace of unobservable state, where θ is given by:

$$\theta = \bigcap_{i=1}^{n} (\text{Kernel } (CA^{i-1}))$$

then by Caley-Hamilton $A\theta \subset \theta$, i.e. θ is also A-invariant.

Such subspaces play an important role in determining the internal structure of linear dynamical systems. For example if $R_o \oplus S_o = X$, and $AR_o \subset R_o$ then it is possible to choose a basis (see Lemma 3.1, in this Appendix) such that $\sigma(\bar{A}) = \sigma(A/R_o)$ correspond to the eigenvalues of the controllable modes and $\sigma(A/R_o)$ correspond to the eigenvalues of the uncontrollable modes (cf. Kalman Canonical form, [A.3]).

A.3 (A,B)-Invariant Subspace

The idea of an A-invariant subspace discussed earlier can be generalized to take into account the effect of state feedback as follows. For a state feedback controller, K, the closed-loop system dynamics are described by the equation:

$$\dot{x}(t) = (A + BK)x(t)$$

A subspace $V \subset X$ is called *(A,B)-invariant* if it is (A + BK)-invariant, that is $(A + BK)V \subset V$. One important point about V being (A + BK)-invariant is the

following: There exists a K such that $(A + BK)\mathcal{V} \subset \mathcal{V}$ if and only if $A\mathcal{V} \subset \mathcal{B} + \mathcal{V}$. One obvious example of (A,B)-invariant subspace is the space spanned by the closed-loop system eigenvectors.

The main usefulness of (A,B)-invariant subspaces arises out of the fact that many control problems can be cast interms of (A,B)-invariant subspaces. Once this is done, the solution for K can be obtained since (A,B)-invariant subspaces can be made (A + BK)-invariant by a suitable choice of K. For example if $\mathcal{V}$ has been constructed to satisfy some conditions then a suitable K can be computed as follows:

$$A\mathcal{V} \subset \mathcal{V} + \mathcal{B} \Rightarrow AV = [V \mid B]R, \qquad R = [V \mid B]^{*}AV$$

where V is now a basis matrix of $\mathcal{V}$, i.e. the columns of V are linearly independent and they span $\mathcal{V}$, and the superscript * denotes the pseudoinverse. If now $[V \mid B]^{*}$ is partitioned as

$$[V \mid B]^{*} = \begin{bmatrix} L_1 \\ \hline L_2 \end{bmatrix}$$

so that the dimensions of L_1 and L_2 are comparable with those of V and B, then

$$AV = [V \mid B]\begin{bmatrix} L_1 \\ \hline L_2 \end{bmatrix}AV = VL_1AV + BL_2AV$$

The above equation can be written as:

$$(A - BL_2A)V = VL_1AV$$

comparing this to the definition for $\mathcal{V}$ as: $(A + BK)\mathcal{V} \subset \mathcal{V}$ yields, K as shown below:

$$K = -L_2A$$

Further discussion on this and related topics can be found in Wonham [A.4], Basile and Marro [A.5] and Morse [A.6].

Proof of Lemma 3.1

Lemma 3.1

Let $\mathcal{M}$ be a nonempty set of nxn matrices in $\mathcal{R}^n$; if $\mathcal{W}$ is an invariant subspace ($\mathcal{W} \subset \mathcal{R}^n$) under $M \in \mathcal{M}$ and is of dimension (n-k) over $\mathcal{R}$, then there exists a basis of $\mathcal{R}^n$ over $\mathcal{R}$ such that every $M \in \mathcal{M}$ in this basis is of the form

$$\begin{bmatrix} M_1 & 0 \\ M_3 & M_4 \end{bmatrix} \qquad \text{(A.1)}$$

where $M_1 \in \mathcal{R}^{k\mathrm{x}k}$ and $M_4 \in \mathcal{R}^{(n-k)\mathrm{x}(n-k)}$. Furthermore, in this basis M_4 is the matrix representation of the restriction of M to $\mathcal{W}$ and M_1 is the restriction of M to $\mathcal{R}^n/\mathcal{W}$.

Proof: If dimension $(\mathcal{X})=n$ then let $\mathcal{X} = \mathcal{W} \oplus \mathcal{Z}$. Let $(z_1, z_2, \ldots, z_k)$ be a basis for $\mathcal{Z}$ and $(w_{k+1}, w_{k+2}, \ldots, w_n)$ be a basis for $\mathcal{W}$. Then $(z_1, z_2, \ldots, z_k, w_{k+1}, w_{k+2}, \ldots, w_n)$ is a basis for $\mathcal{X}$. It is clear that

$$z \in \mathcal{Z} \to z = \sum_{i=1}^{k} \alpha_i z_i \tag{A.2}$$

and that

$$w \in \mathcal{W} \to w = \sum_{i=k+1}^{n} \alpha_i w_i \tag{A.3}$$

Thus when z_i and w_i are operated on by M, it is easy to see that:

$$Mz_j = \sum_{i=1}^{k} m_{ij} z_i + \sum_{i=k+1}^{n} m_{ij} w_i \tag{A.4}$$

and
$$Mw_j = \sum_{i=1}^{k} m_{ij} z_i + \sum_{i=k+1}^{n} m_{ij} w_i \tag{A.5}$$

In eqn. (A.5) the elements m_{ij} are the elements of the matrix representation of M to $\mathcal{W}$. Because $\mathcal{W}$ is M-invariant, in eqn. (A.5), $m_{ij} = 0$ for $i = 1,k$ and $j = k+1,n$ $(\to M_2 = 0)$. This means that M in the basis $(z_1, z_2, \ldots, z_k, w_{k+1}, \ldots, w_n)$ is of the form:

$$\begin{bmatrix} M_1 & 0 \\ M_3 & M_4 \end{bmatrix}$$

and that M_4 is the matrix representation of the restriction of M to $\mathcal{W}$. Also $[M_1\ M_3]^T$ is the matrix representation of the restriction of M to $\mathcal{Z}$ and since $\mathcal{W}$ represents the zero element in the quotient space $\mathcal{R}^n/\mathcal{W}$, M_1 is the matrix representation of the restriction to $\mathcal{R}^n/\mathcal{W}$; or M_1 is the matrix of the linear map on $\mathcal{R}^n/\mathcal{W}$ induced by M (in the basis $\{(z_1 + \mathcal{W}), (z_2 + \mathcal{W}), \ldots\}$).

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APPENDIX B

EIGENVALUE/EIGENVECTOR ASSIGNMENT USING OUTPUT FEEDBACK CONTROL

This appendix is in support of the material covered in Chapter 2. The following discussion outlines the algebraic manipulations required to arrive at an algorithm for eigenvalue/eigenvector assignment using output feedback. It also points out the difficulties associated with the proposed algorithm.

Consider a linear nth order multivariable system with r inputs and m outputs as given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \qquad \text{(B.1)}$$

and subject to the output feedback control law

$$u(t) = Ky(t) = KCx(t) \qquad \text{(B.2)}$$

The resulting closed-loop system matrix is given by:

$$(A + BKC) = WJV \qquad \text{(B.3)}$$

Partitioning eqn. (B.3) gives

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] K \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] = \left[\begin{array}{c|c} W_{11} & W_{12} \\ \hline W_{21} & W_{22} \end{array}\right] \left[\begin{array}{c|c} J_1 & 0 \\ \hline 0 & J_2 \end{array}\right] \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{21} & V_{22} \end{array}\right]$$

Assuming r = m, then in the above equation A_{11}, B_1, C_1, W_{11}, J_1 and V_{11} are mxm matrices; A_{22}, W_{22}, J_2 and V_{22} are (n-m)x (n-m) matrices; K is mxm and B_1 and C_1 are assumed to be non-singular, obtained if necessary by reordering of the state variables.

From eqn. (B.3):

$$A_{11} + B_1KC_1 = W_{11}J_1V_{11} + W_{12}J_2V_{21} \tag{B.4}$$

$$A_{21} + B_2KC_1 = W_{21}J_1V_{11} + W_{22}J_2V_{21} \tag{B.5}$$

and hence

$$B_1K = (W_{11}J_1V_{11} + W_{12}J_2V_{21} - A_{11})C_1^{-1} \tag{B.6}$$

and $$B_2K = (W_{21}J_1V_{11} + W_{22}J_2V_{21} - A_{21})C_1^{-1} \tag{B.7}$$

Also

$$A_{22} + B_2KC_2 = W_{21}J_1V_{12} + W_{22}J_2V_{22} \tag{B.8}$$

and $$A_{12} + B_1KC_2 = W_{11}J_1V_{12} + W_{12}J_2V_{22} \tag{B.9}$$

Substituting (B.7) into (B.8) gives:

$$A_{22} + (W_{21}J_1V_{11} + W_{22}J_2V_{21} - A_{21})C_1^{-1}C_2 = W_{21}J_1V_{12} + W_{22}J_2V_{22} \qquad \ldots\text{(B.10)}$$

and substituting (B.6) into (B.9) gives

$$A_{12} + (W_{11}J_1V_{11} + W_{12}J_2V_{21} - A_{11})C_1^{-1}C_2 = W_{11}J_1V_{12} + W_{12}J_2V_{22} \qquad \ldots\text{(B.11)}$$

Let $Q = C_1^{-1}C_2$. Premultiply (B.11) by V_{11}:

$$V_{11}W_{11}J_1V_{11}Q + V_{11}W_{12}J_2V_{21}Q - V_{11}A_{11}Q = V_{11}W_{11}J_1V_{12} + V_{11}W_{12}J_2V_{22} - V_{11}A_{12} \tag{B.12}$$

using the identity $V_{11}W_{11} + V_{12}W_{21} = I$ gives

$$J_1V_{11}Q - V_{11}A_{11}Q + V_{11}A_{12} - J_1V_{12} = V_{12}W_{21}J_1V_{11}Q - V_{11}W_{12}J_2V_{21}Q - V_{12}W_{21}J_1V_{12} + V_{11}W_{12}J_2V_{22} \tag{B.13}$$

Pre-multiplying (B.10) by V_{12} gives

$$V_{12}A_{22} - V_{12}A_{21}Q = V_{12}W_{21}J_1V_{12} + V_{12}W_{22}J_2V_{22} - V_{12}W_{21}J_1V_{11}Q - V_{12}W_{22}J_2V_{21}Q \quad \cdot\cdot \text{(B.14)}$$

Using the identity $V_{11}W_{12} = -V_{12}W_{22}$ gives

$$V_{12}A_{22} - V_{12}A_{21}Q = V_{12}W_{21}J_1V_{12} - V_{11}W_{12}J_2V_{22} - V_{12}W_{21}J_1V_{11}Q + V_{11}W_{12}J_2V_{21}Q \quad \cdot\cdot \text{(B.15)}$$

Substituting (B.15) into (B.13) gives

$$J_1V_{12} - V_{12}(A_{22} - A_{21}Q) = J_1V_{11}Q + V_{11}(A_{12} - A_{11}Q) \quad \text{(B.16)}$$

Let $L = A_{12} - A_{11}Q$ and $R = A_{22} - A_{21}Q$

Then (B.16) becomes

$$\boxed{J_1V_{12} - V_{12}R = J_1V_{11}Q + V_{11}L} \quad \text{(B.17)}$$

Premultiplying eqn. (B.10) by V_{22} gives

$$V_{22}A_{22} - V_{22}A_{21}Q = V_{22}W_{21}J_1V_{12} + V_{22}W_{22}J_2V_{22} - V_{22}W_{21}J_1V_{11}Q - V_{22}W_{22}J_2V_{21}Q \quad \cdot\cdot \text{(B.18)}$$

Using the identity $V_{21}W_{11} = -V_{22}W_{21}$ gives

$$V_{22}A_{22} - V_{22}A_{21}Q = -V_{21}W_{11}J_1V_{12} + V_{22}W_{22}J_2V_{22} + V_{21}W_{11}J_1V_{11}Q - V_{22}W_{22}J_2V_{21}Q \quad \cdot\cdot \text{(B.19)}$$

Premultiplying eqn. (B.11) by V_{21} gives

$$V_{21}A_{12} - V_{21}A_{11}Q + V_{21}W_{11}J_1V_{11}Q + V_{21}W_{12}J_2V_{21}Q = V_{21}W_{11}J_1V_{12} + V_{21}W_{12}J_2V_{22} \quad \cdot\cdot \text{(B.20)}$$

Using the identity $V_{21}W_{12} = I - V_{22}W_{22}$ gives

$$V_{21}A_{12} - V_{21}A_{11}Q + J_2V_{21}Q - J_2V_{22} = V_{21}W_{11}J_1V_{12} - V_{22}W_{22}J_2V_{22} - V_{21}W_{11}J_1V_{11}Q + V_{22}W_{22}J_2V_{21}Q \quad \text{(B.21)}$$

substituting (B.19) into (B.21) gives

$$J_2V_{22} - V_{22}(A_{22} - A_{21}Q) = J_2V_{21}Q + V_{21}(A_{12} - A_{11}Q) \tag{B.22}$$

Using expressions for L and R gives:

$$\boxed{J_2V_{22} - V_{22}R = J_2V_{21}Q + V_{21}L} \tag{B.23}$$

From eqn. (B.4)

$$KC_1 = B_1^{-1}(W_{11}J_1V_{11} + W_{12}J_2V_{21} - A_{11}) \tag{B.24}$$

and from eqn. (B.9)

$$KC_2 = B_1^{-1}(W_{11}J_1V_{12} + W_{12}J_2V_{22} - A_{12}) \tag{B.25}$$

Substituting (B.24) into (B.5)

Let $S = B_2B_1^{-1}$

$$A_{21} + SW_{11}J_1V_{11} + SW_{12}J_2V_{21} - SA_{11} = W_{21}J_1V_{11} + W_{22}J_2V_{21} \tag{B.26}$$

Substituting (B.25) into (B.8)

$$A_{22} + SW_{11}J_1V_{12} + SW_{12}J_2V_{22} - SA_{12} = W_{21}J_1V_{12} + W_{22}J_2V_{22} \tag{B.27}$$

Post-multiplying (B.26) by W_{11} and (B.27) by W_{21} gives:

$$A_{21}W_{11} - SA_{11}W_{11} + SW_{11}J_1(I - V_{12}W_{21}) = W_{21}J_1(I - V_{12}W_{21}) + W_{22}J_2V_{21}W_{11} - SW_{12}J_2V_{21}W_{11} \tag{B.28}$$

which on simplification gives:

$$A_{21}W_{11} - SA_{11}W_{11} + SW_{11}J_1 - W_{21}J_1 = SW_{11}J_1V_{12}W_{21} - W_{21}J_1V_{12}W_{21} + W_{22}J_2V_{21}W_{11} - SW_{12}J_2V_{21}W_{11} \tag{B.29}$$

and

$$A_{22}W_{21} - SA_{12}W_{21} = W_{21}J_1V_{12}W_{21} + W_{22}J_2V_{22}W_{21} - SW_{11}J_1V_{12}W_{21} - SW_{12}J_2V_{22}W_{21} \tag{B.30}$$

using the identify $V_{21}W_{11} = -V_{22}W_{21}$

$$A_{22}W_{21} - SA_{12}W_{21} = -SW_{11}J_1V_{12}W_{21} + W_{21}J_1V_{12}W_{21} - W_{22}J_2V_{21}W_{11} + SW_{12}J_2V_{21}W_{11} \qquad \text{(B.31)}$$

substituting (B.31) into (B.29) gives:

$$A_{21}W_{11} - SA_{11}W_{11} + SW_{11}J_1 - W_{21}J_1 = SA_{12}W_{21} - A_{22}W_{21}$$

that is

$$W_{21}J_1 - (A_{22} - SA_{12})W_{21} = SW_{11}J_1 + (A_{21} - SA_{11})W_{11} \qquad \text{(B.32)}$$

Letting $T = A_{21} - SA_{11}$

and $P = A_{22} - SA_{12}$

Eqn. (B.32) can be rewritten as:

$$\boxed{W_{21}J_1 - PW_{21} = SW_{11}J_1 + TW_{11}} \qquad \text{(B.33)}$$

In the same manner post multiplying (B.26) by W_{12} and (B.27) by W_{22} gives:

$$\boxed{W_{22}J_2 - PW_{22} = TW_{12} + SW_{12}J_2} \qquad \text{(3.34)}$$

Equations (B.17), B.23),(B.33) and (B.34) are relevant for the design procedure. Inspection of eqns. (B.7) and (B.33) shows that these two equations have to be solved iteratively with the identity: $V_{11}W_{11} + V_{12}W_{21} = I$. That is, having made an arbitrary choice of W_{11} and V_{11}, W_{21} and V_{12} are evaluated from eqns. (B.17) and (B.33) respectively, and a check is made to see if the identity $V_{11}W_{11} + V_{12}W_{21} = I$ is satisfied; if not, new choices are made for initial

values of W_{11} and V_{11} and the procedure repeated. Clearly convergence cannot be guaranteed for such a procedure and it becomes at once evident that the procedure can be quite lengthy. Once an appropriate choice of W_{11} and V_{11} is made and W_{21} and V_{12} have been computed such that the required identity is satisfied, then a choice of W_{12} and V_{21} is made and eqns. (B.23) and (B.34) are solved for V_{22} and W_{22}, respectively. Now a test is made to determine whether the following identities are satisfied: $V_{11}W_{12} + V_{12}W_{22} = 0$, $V_{21}W_{11} + V_{22}W_{21} = 0$ and $V_{21}W_{12} + V_{22}W_{22} = I$. If not, the procedure is repeated for different choices of W_{12} and V_{21}. It is thus clear that the algorithm can be extremely lengthy and that there are considerable difficulties involved with the design procedure for eigenvalue/eigenvector assignment using output feedback.

Note that the procedure outlined briefly at the end of Reference B.1 may at first appear to be slightly different than the one above. However, both are equivalent. By initially choosing only W_{11} and V_{11} arbitrarily then computing W_{21} and V_{12} (from eqns. (B.17) and (B.33)) and finally checking to see if the identity $V_{11}W_{11} + V_{12}W_{21} \equiv I$ holds, the procedure as proposed above should be slightly easier to follow. Note that convergence,

i.e. getting the correct values of V and W such that $VW \equiv I$ holds as well as eqns. (B.17), (B.23), (B.33) and (B.34), is not guaranteed.

This appendix essentially points out the lengthy and difficult design procedure that follows as a result of difficulties imposed by the rather restrictive set of necessary and sufficient conditions for eigenvalue/eigenvector assignment using output feedback as outlined in Chapter 2.

REFERENCE FOR APPENDIX B

[B.1] Shah, S.L., Fisher, D.G., and Seborg, D.E., 'Eigenvalue/eigenvector assignment for multivariable systems and further results for output feedback control', Electronics Lett., 1975, 11, pp. 388-389.

APPENDIX C

PILOT PLANT EVAPORATOR AND ITS MODELS

C.1 The Evaporator

The pilot plant evaporator used in this work is a double effect unit with the two effects operating in series. This pilot plant has been described in detail by Fisher and Seborg [C.1]. The major pieces of process equipment are shown in the schematic diagram in Figure C.1. The control loops shown in Figure C.1 represent the multiloop control scheme applied to the evaporator in previous studies [C.1].

The first effect has natural circulation through its 18 inch long, 3/4 inch O.D. tubes. It is heated by process steam. The second effect is a long tube vertical unit which is run in its forced circulation mode. It has three, six foot long, one inch O.D. tubes. It is operated at a lower pressure than the first effect and is heated by the vapour produced in the first effect.

The evaporator is fully instrumented and can be controlled by either Foxboro electronic controllers or under Direct Digital Control (DDC) from an IBM 1800 Data Acquisition and Control Computer operating under MPX. Multiloop DDC can be applied directly using the computer

control package and advanced control schemes by user written programs which utilize a set of system programs to interface between the user and system control programs.

C.2 The Evaporator Model

The complete development of the double-effect evaporator model has been presented by Newell [C.2] who derived fifth and tenth order non-linear models. Based on these models Wilson [C.3] calculated discrete five-state and three-state, linear, time-invariant models using a linearization procedure and Marshall's model reduction method.

The linearized models in the discrete form can be represented by:

$$x(n+1) = \Phi x(n) + \Delta u(n) + \Theta d(n) \tag{C.1}$$

and

$$y(n) = C\, x(n) \tag{C.2}$$

The elements of the vectors x, u, d, y are defined as normalized perturbation variables:

$$x_1 = \frac{W_1 - W_{1ss}}{W_{1ss}} \tag{C.3}$$

where W_{1ss} is the normal steady state value of W_1.

The vectors x, u, d and y for the fifth order discrete model are defined in Table C.1. The coefficient matrices of this discrete time model, with a 64 second sampling interval, are shown in Table C.2.

For the third order discrete model the state vector x is given by:

$$x = (W1, W2, C2)^T \tag{C.4}$$

The elements of the state vector are defined in Table C.1. The vectors u, d and y are equal to the ones defined in Table C.1. The coefficient matrices for this model, with a 64 second sampling interval, are shown in Table C.3.

REFERENCES FOR APPENDIX C

[C.1] Fisher, D.G., and Seborg, D.E., Multivariable Computer Control - A Case Study, American Elsevier Publ. Co., N.Y., 205 pp, 1976.

[C.2] Newell, R.B., 'Multivariable computer control of an evaporator', Ph.D. Thesis, Dept. of Chemical Eng., U. of Alberta, 1971.

[C.3] Wilson, R.G., 'Model reduction and reduced order control law design', Ph.D. Thesis, Dept. of Chemical Eng., U. of Alberta, 1974.

TABLE C.1

DESCRIPTION OF THE EVAPORATOR VARIABLES

State Vector, x		Normal Steady State Value
$x^T = [W1, C1, H1, W2, C2]$		
W1	First effect holdup	45.5 lb.
C1	First effect concentration	4.59% glycol
H1	First effect enthalpy	189.2 BTU/lb.
W2	Second effect holdup	41.5 lb.
C2	Second effect concentration	10.11% glycol

Control Vector, u		
$u^T = [S, B1, B2]$		
S	Steam flow	2.0 lb./min.
B1	First effect bottoms flow	3.485 lb./min.
B2	Second effect bottoms flow	1.581 lb./min.

Disturbance Vector, d		
$d^T = [F, CF, HF]$		
F	Feed flow	5.0 lb./min.
CF	Feed concentration	3.2% glycol
HF	Feed enthalpy	156.9 BTU/lb.

Output Vector, y

$y^T = [W1, W2, C2]$

TABLE C.2

FIFTH ORDER DISCRETE EVAPORATOR MODEL (T = 64 sec.)

$$\phi = \begin{bmatrix} 1.0 & -0.0008 & -0.0912 & 0 & 0 \\ 0 & 0.9223 & 0.0871 & 0 & 0 \\ 0 & -0.0042 & 0.4376 & 0 & 0 \\ 0 & -0.0009 & -0.1052 & 1.0 & 0.0001 \\ 0 & 0.0391 & 0.1048 & 0 & 0.9603 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} -0.0119 & -0.0817 & 0 \\ 0.0116 & 0 & 0 \\ 0.0116 & 0 & 0 \\ -0.0138 & 0.0848 & -0.0406 \\ 0.0137 & -0.0432 & 0 \end{bmatrix} \qquad \theta = \begin{bmatrix} 0.1182 & 0 & -0.0050 \\ -0.0351 & 0.0785 & 0.0049 \\ -0.0136 & -0.0002 & 0.0662 \\ 0.0012 & 0 & -0.0058 \\ 0.0019 & 0.0016 & 0.0058 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

TABLE C.3

THIRD ORDER DISCRETE EVAPORATOR MODEL (T = 64 sec.)

$$\phi = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 0.9602 \end{bmatrix} \qquad \Delta = \begin{bmatrix} -0.0326 & -0.0811 & 0 \\ 0.0378 & 0.0854 & -0.0406 \\ 0.0529 & -0.0442 & 0 \end{bmatrix}$$

$$\theta = \begin{bmatrix} 0.1200 & 0 & -0.0135 \\ 0.0033 & 0 & -0.0157 \\ -0.0219 & 0.0400 & 0.0219 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

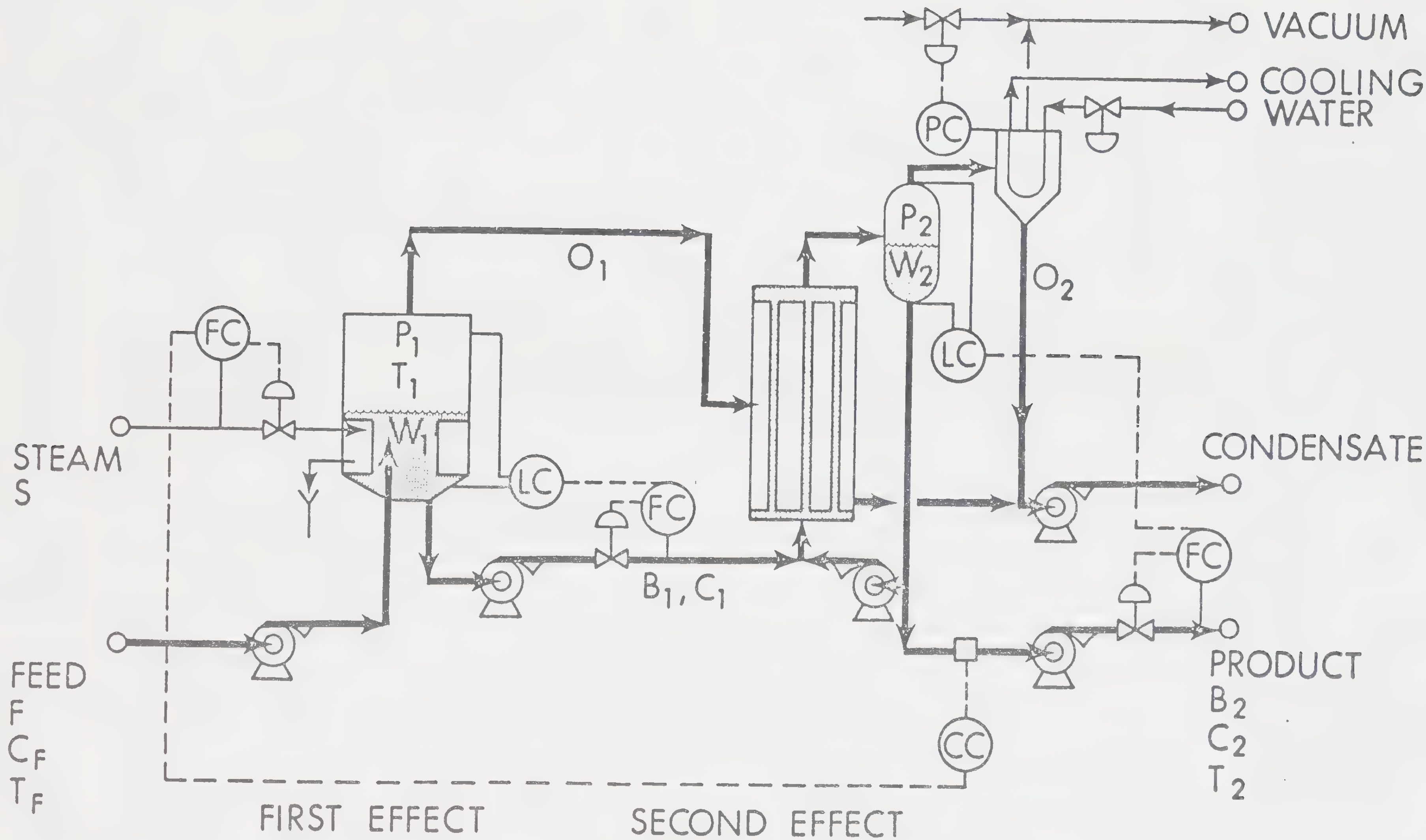

FIGURE C.1: Schematic Diagram of the Double Effect Pilot Plant Evaporator used for this work.

APPENDIX D

ON INVARIANT ZEROS, INVARIANT SUBSPACES AND PARAMETRIC SENSITIVITY

D.1 Introduction

This Appendix extends the discussion of Chapters 3 and 6 and is particularly concerned with relating the following three relatively new concepts: (i) invariant zeros (as defined by Bengtsson [D.1]), (ii) (A,B)-invariant subspaces [D.2] (see also Appendix A), and (iii) eigenvalue insensitivity to variations in system parameters [D.3](see also Chapter 6).

In [D.1] the invariant zeros of a MIMO system have been defined as those zeros* of a system that are invariant under state feedback in the sense that the open-loop system denoted by S(A,B,C) and the closed-loop system S(A+BK,B,C) have the same set of zeros for a class of linear maps K. The concept of (A,B)-invariant subspace has played a key role in a number of control problems [D.2]. The most familiar one is the disturbance localisation problem. In the following discussion these two concepts will be related. It will be shown that when certain set of conditions are imposed on (A,B)-invariant subspaces, then a simple algorithm to compute the invariant zeros of

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* for the general definition of zeros of multivariable systems see Macfarlane and Karcanias [D.4].

a (left) invertible system can be derived. In order to show a use for such an algorithm we will relate invariant zeros to insensitive eigenvalues. In particular, it will be shown that the invariant zeros are the same as those eigenvalues which are insensitive or invariant to *arbitrary* parameter variations in specified columns of the A or K matrices. In contrast to results in the first part of Chapter 6 which are strictly valid only for eigenvalue invariance or insensitivity to *small*, *arbitrary* perturbations in specified columns of A or K, it will be shown here that the analysis carried out here allows a designer to make certain eigenvalues invariant to unknown variations of arbitrary magnitude in specified columns of A or K (Cf. Theorem 6.1).

D.2 Invariant Zeros and Invariant Subspaces

In [D.2] it is shown that the maximal (A,B)-invariant subspace, $\mathcal{V}^m$, contained in Kernel C, can be obtained by the following sequence (see also discussion of (A,B) - invariant subspaces in Appendix A).

$$\mathcal{V}_o = \text{Kernel } C$$

$$\mathcal{V}_i = \mathcal{V}_{i-1} \cap A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}) \quad i = 1,2,....n$$

where $\mathcal{B}$ denotes Range B and

$$A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}) = \{x \in \mathcal{R}^n \mid A\,x \in (\mathcal{V}_{i-1} + \mathcal{B})\}.$$

Then let p be the first integer such that $\mathcal{V}_{p+1} = \mathcal{V}_p$, then $\mathcal{V}^m = \mathcal{V}_p$. This sequence converges after at most ν steps, where ν = dim (Kernel C).

Note that to simplify the calculation of $\mathcal{V}^m$ we impose the following condition on $(\mathcal{V}^o + \mathcal{B})$, i.e. assume that dimension $(\mathcal{V}^o + \mathcal{B}) = n$. Then

$$\mathcal{V}_1 = \mathcal{V}_o \cap A^{-1}(\mathcal{V}^o + \mathcal{B}) = \mathcal{V}_o = \mathcal{V}^m = \text{Kernel C}.$$

For the case where $k = r$ (i.e. the number of undisturbable state variables or outputs (k) is equal to the number of input or control variables (r)), the condition:

$$\text{dimension } (\mathcal{V}^o + \mathcal{B}) = n,$$

implies the following:

i) The first k rows of B must be linearly independent.

ii) In addition dimension $\mathcal{V}^o = n-k$ since C can be written as $C = [I_k \;\; O]$, and in order to realise the property $(A + BK)\mathcal{V}^o \subset \mathcal{V}^o$ it is required that (n-k) of the eigenvectors of (A + BK) lie in $\mathcal{V}^o$, i.e. (n-k) of the closed-loop eigenvectors lie in Kernel C. Using the eigenvector/eigenvalue assignment technique (of Chapter 2) it becomes clear that for (n-k) of the closed-loop eigenvectors to lie in Kernel C, $W_{12} = O$. The eigenvalues corresponding to these eigenvectors are then given by:

$$PW_{22} = W_{22}\,\Lambda_2$$

where $P = A_{22} - B_2 B_1^{-1} A_{12}$

and the diagonal elements of Λ_2 are the eigenvalues of P.

D.2.1 <u>A new algorithm to compute invariant zeros of a system</u>

Note that for the above case $\sigma(A + BK|\mathcal{V}^m) = \sigma(P)$. From [D.1], invariant zeros of a left invertible system are $\sigma(A + BK|\mathcal{V}^m)$. Hence we have the following new algorithm to compute the invariant zeros of a system:

If dimension $(\mathcal{V}^o + \mathcal{B}) = n$ and $k = r$, i.e. the first k or r rows of B are linearly independent then the invariant zeros of the system are given by the eigenvalues of $(A_{22} - B_2 B_1^{-1} A_{12})$ where A_{22} is the (n-k) x (n-k) lower right matrix partition of A and A_{12} is (k x n-k) upper right matrix partition of A. B_1 and B_2 are the top k x r (note k = r) (or k x k) and bottom (n - k) x r (or (n-k) x k) partitions of B respectively.

<u>Example D.1</u>:

Consider a system with the triple (A,B,C) given by:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

To rewrite the output matrix of the above system as

$C^{\dagger} = [\ I_2 \quad 0]$, use will be made of the transformation

(Cf. Section 3.9) $X = QZ$ where Q is given by:

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$Y = CX = CQZ = C^{\dagger}Z = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

Thus

$$\mathcal{V}_o = \operatorname{Ker} C^{\dagger} = \left\{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right\} = \mathcal{V}^m$$

$$A^{\dagger} = Q^{-1} AQ = \left[\begin{array}{cc|c} -1 & 0 & -2 \\ 0 & -1 & 0 \\ \hline 0 & 0 & -3 \end{array}\right] \triangleq \left[\begin{array}{c|c} A_1^{\dagger} & A_2^{\dagger} \\ \hline A_3^{\dagger} & A_4^{\dagger} \end{array}\right]$$

$$B^{\dagger} = Q^{-1} B = \left[\begin{array}{cc} 1 & 2 \\ 1 & 1 \\ \hline 0 & 2 \end{array}\right] = \left[\begin{array}{c} B_1^{\dagger} \\ \hline B_2^{\dagger} \end{array}\right]$$

clearly dimension $(\mathcal{V}_o + \mathcal{B})$ is equal to

$$\text{dimension} \left\{ \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}, \begin{matrix} 1 \\ 1 \\ 0 \end{matrix}, \begin{matrix} 2 \\ 1 \\ 2 \end{matrix} \right\} = 3$$

and with first two rows of $B^{\dagger}$ linearly independent invariant zeros of the above system are given by

$$= \sigma\ (A_4^{\dagger} - B_2^{\dagger} B_1^{\dagger -1}\ A_2^{\dagger})$$

$$= \sigma\ \left(-3-[0\ \ 2] \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix}\right)$$

$$= \sigma\ (-3+4) = \{1.0\}$$

<u>Example D.2</u>

Consider the SISO system with the triple (A,B,C) given by:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad C = [1 \quad -1 \quad 1]$$

using the transformation (Cf. Section 3.9) $X = QZ$

$$\text{where} \qquad Q = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C^{\dagger} = CQ = [1 \mid 0 \quad 0]$$

$$A^{\dagger} = Q^{-1}AQ = \left[\begin{array}{c|cc} -1 & 0 & -1 \\ \hline 0 & -1 & 0 \\ 0 & 0 & -2 \end{array}\right] \triangleq \left[\begin{array}{c|c} A_1^{\dagger} & A_3^{\dagger} \\ \hline A_3^{\dagger} & A_4^{\dagger} \end{array}\right]$$

$$B^{\dagger} = Q^{-1}B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \triangleq \begin{bmatrix} B_1^{\dagger} \\ B_2^{\dagger} \end{bmatrix}$$

Now $\mathcal{V}_o = \text{Ker } C = \left\{ \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} , \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\}$, also dim. $(\mathcal{V}_o + \mathcal{B}) = 3$

and since the first row ($k = 1$) of $B^{\dagger}$ is linearly independent the invariant zeros of the above system are

$$= \sigma\ (A_4^{\dagger} - B_2^{\dagger} B_1^{\dagger -1}\ A_2^{\dagger})$$

i.e. eigenvalues of $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} [0 \quad -1]$

$$= \sigma \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \{-1\ ,\ -1\}$$

Note that these are precisely the eigenvalues which cannot be assigned arbitrarily in the design of a state feedback controller to make $z_1 (=y_1)$ undisturbable.

D.3 Invariant Zeros and Parametric Insensitivity

For the class of systems considered here, namely where $C = [I_k \quad O]$, the first k rows of B are linearly independent and k = r, and hence $\mathcal{V}_o = \mathcal{V}^m$ = Kernel C, it is clear that in order to satisfy $(A + BK)\mathcal{V}^o \subset \mathcal{V}^o$, (n-k) of the closed-loop system eigenvectors must lie in Kernel C. In other words the first k elements of the (n-k) eigenvectors must be zero, i.e. $W_{12} = 0$. For such a class of systems there are (n-k) invariant zeros.

For such cases it can be shown that:

i) it is possible to make as many eigenvalues invariant or insensitive to parameter variations (in specified columns of A or K), as the number of invariant zeros of the system, and

ii) the invariant or insensitive eigenvalues remain unmoved despite arbitrary and unknown parameter changes in specified columns of K and hence the A matrix. Note also that these parameter variations need not necessarily be small. This point will be clarified shortly.

For a left invertible system {invariant or insensitive eigenvalues} = $\sigma((A + BK)|\mathcal{V}_o)$. Denote by K^m the class of all feedback matrices K such that $K_o \in K^m$ satisfies

$(A + BK_o)V_o \subset V_o$, that is $K^m = \{K | (A + BK)V_o \subset V_o\}$.

Now for $C = [I_K | O_{n-K}]$ and $V_o = \text{Ker } C$,

$$V_o = \left\{ \begin{array}{c} O_K \\ \hline I_{n-K} \end{array} \right\} \quad \text{where } \dim V_o = n-K.$$

Then for such a class of (A,B)-invariant subspaces - i.e. V_o, it can be shown that

$$K^m = \{[K_i \,\vdots\, K_o]\} \qquad i = 1,2,\ldots\ldots.$$

where K_i is a variable $k \times k$ matrix (assume $k = r$) and K_o is a $k \times (n-k)$ matrix of constants. In other words *for any arbitrary variation* in K_i (i.e. the first k columns of K) or equivalently in the first k columns of A, the eigenvalues $\sigma(A + BK|V^o)$ remain invariant or insensitive. This point will be illustrated by the following example:

<u>Example D.3</u>

Consider the system S(A,B,C) where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then $V_o = \text{Kernel } C = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$.

The open-loop eigenvalues are 0, 1, and -1. If the measurements of x_1 and x_2 are unreliable or the parameters in the first and second columns of A are likely to vary then we can design a K such that $(A + BK)\mathcal{V}_o \subset \mathcal{V}_o$, with the resulting invariant eigenvalue of (-1) being completely insensitive to *any arbitrary variation* in the first two columns of K or A. The remaining two eigenvalues of $(A + BK)$ can be assigned arbitrarily.

Suppose then W_{11} is chosen as

$$W_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad W_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(for invariance to first and second columns of A and K)

and $$\Lambda_1 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} .$$

Then Λ_3 turns out to be $\sigma(P) = -1$ and

$W_{21} = [2 \quad 0]$ and $W_{22} = [1]$.

Thus $$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and } J = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

Then $$K = B^*(WJ\mathcal{V}-A) = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -4 & -1 \end{bmatrix} .$$

Where B^* is the pseudoinverse of B.

This K satisfies $(A + BK)\mathcal{V}_o \subset \mathcal{V}_o$. Let K^m be a class of all K such that $(A + BK)\mathcal{V}_o \subset \mathcal{V}_o$ then rewrite $K \varepsilon K^m$ as $K = [K_i \; K_o]$, where K_i is 2×2 matrix and $K_o = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Then it can be seen easily that for any arbitrary variations in K_i or A_1, or A_2 (first and second columns of A) the eigenvalue, -1, remains unmoved. For example if

$$K' = \left[\begin{array}{cc|c} -1 & -\frac{1}{2} & 0 \\ 0 & -2 & -1 \end{array}\right]$$

corresponding to 50% error in the transducer gain for x_1 and x_2, then

$$A + BK' = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 \\ -1 & \frac{1}{2} & -1 \end{bmatrix}$$

i.e. the eigenvalue (-1) remains unmoved. Likewise if

A_1 becomes $[1\;1\;1]^T$ and A_2 becomes $[1.2\;1.6\;1.4]^T$, then

$$A' + BK = \begin{bmatrix} 1 & 1.2 & 0 \\ 1 & 1.6 & 1 \\ 1 & 1.4 & -1 \end{bmatrix} + \begin{bmatrix} -2 & -1 & 0 \\ 0 & -4 & -1 \\ -2 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0.2 & 0 \\ 1 & -2.4 & 0 \\ -1 & 0.4 & -1 \end{bmatrix}$$

and the resulting eigenvalues are: -1, -2.2 and -1.2, where the invariant eigenvalue is -1. Notice that the variations in K and A are of *significant magnitude* and usually cannot be handled by conventional sensitivity techniques.

Another result that follows easily from the above analysis is that the initial condition response of the above system for all $x(o) \in V_o$ is invariant regardless of any variations in the first two columns of A or K. This corresponds to *trajectory invariance* of all $x(t)$ for all initial conditions $x(o)$ which lie in V_o. (Cf. Remark 6.3). For example for the above case the response of $x_1(t)$, $x_2(t)$ and $x_3(t)$ is unchanged for $x(o)^T = [0 \quad 0 \quad \alpha] \; \forall \; \alpha \neq 0$, i.e. $x(o) \in V^o$. Also for $b_j \in V^o$ (b_j is the jth column of B) the class of K such that $(A + BK)V^o \subset V^o$ results in trajectory invariance. (This result is really related to disturbance localisation.)

D.4 Concluding Remarks

In this Appendix the concepts of invariant zeros, invariant subspaces and parametric insensitivity have been related. The three illustrative examples show the use of the proposed algorithm for calculation of invariant zeros of a system. It is also shown how the invariant eigenvalues are the same as invariant zeros of the

system, and hence are insensitive to arbitrarily large and unknown parameter variations in specified columns of the A or K matrices. For single-input single-output systems the relationship between invariant zeros and parametric insensitivity can also be understood from a root-loci point of view - that is because of a pole-zero cancellation (invariant eigenvalue - invariant zero cancellation) variation in certain elements of K does not move the invariant eigenvalue.

REFERENCES FOR APPENDIX D

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